Why Sorting?

- “When in doubt, sort” – one of the principles of algorithm design. Sorting used as a subroutine in many of the algorithms:
  - Searching in databases: we can do binary search on sorted data
  - A large number of computer graphics and computational geometry problems
  - Closest pair, element uniqueness

Why Sorting? (2)

- A large number of sorting algorithms are developed representing different algorithm design techniques.
- A lower bound for sorting $\Omega(n \log n)$ is used to prove lower bounds of other problems

Sorting Algorithms so far

- Insertion sort, selection sort
  - Worst-case running time $\Theta(n^2)$; in-place
- Heap sort
  - Worst-case running time $\Theta(n \log n)$.

Divide and Conquer

- *Divide-and-conquer* method for algorithm design:
  - **Divide**: if the input size is too large to deal with in a straightforward manner, divide the problem into two or more disjoint subproblems
  - **Conquer**: use divide and conquer recursively to solve the subproblems
  - **Combine**: take the solutions to the subproblems and “merge” these solutions into a solution for the original problem

Merge Sort Algorithm

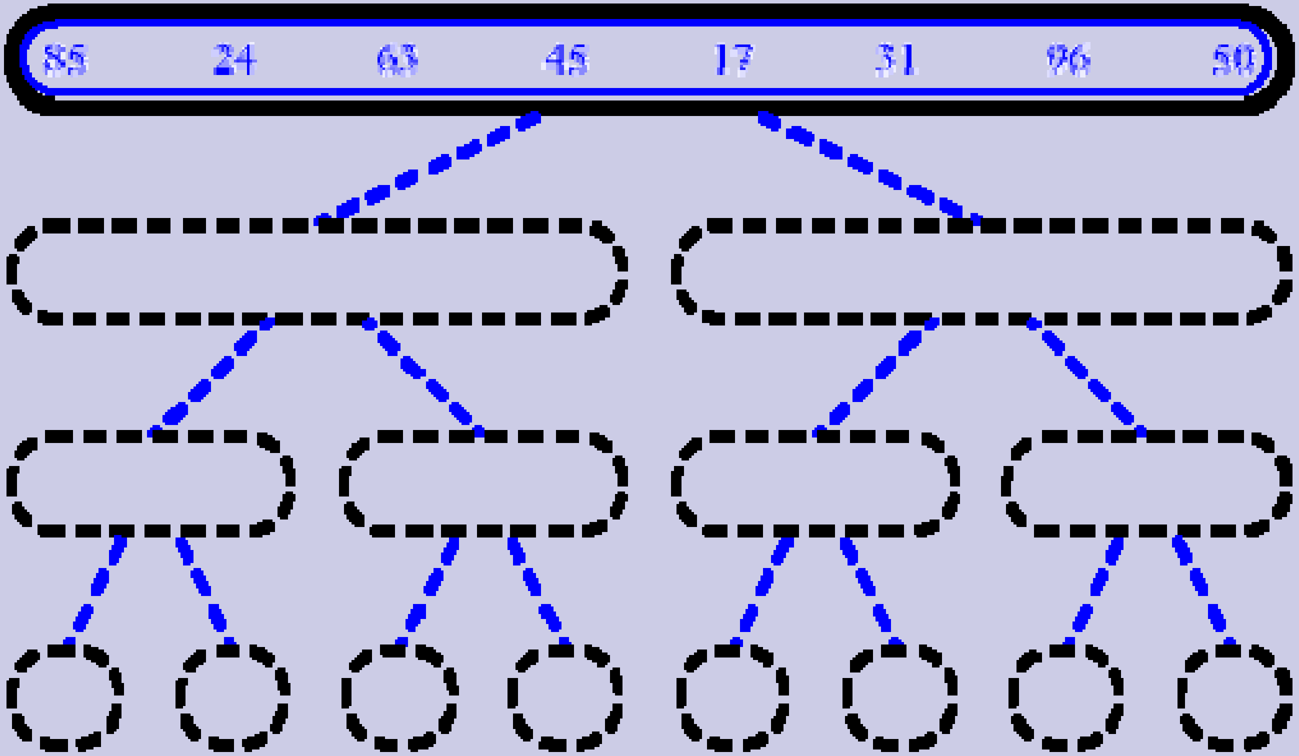
- **Divide**: If *S* has at least two elements (nothing needs to be done if *S* has zero or one elements), remove all the elements from *S* and put them into two sequences, S_1 and S_2 , each containing about half of the elements of *S*. (i.e. S_1 contains the first $\lceil n/2 \rceil$ elements and S_2 contains the remaining $\lfloor n/2 \rfloor$ elements).
- **Conquer**: Sort sequences S_1 and S_2 using Merge Sort.
- **Combine**: Put back the elements into *S* by merging the sorted sequences S_1 and S_2 into one sorted sequence

Merge Sort: Algorithm

```
Merge-Sort(A, p, r)
   if p < r then
      q←(p+r)/2
      Merge-Sort(A, p, q)
      Merge-Sort(A, q+1, r)
      Merge(A, p, q, r)
```

```
Merge(A, p, q, r)
   Take the smallest of the two topmost elements of
sequences A[p..q] and A[q+1..r] and put into the
resulting sequence. Repeat this, until both sequences
are empty. Copy the resulting sequence into A[p..r].
```

MergeSort (Example)



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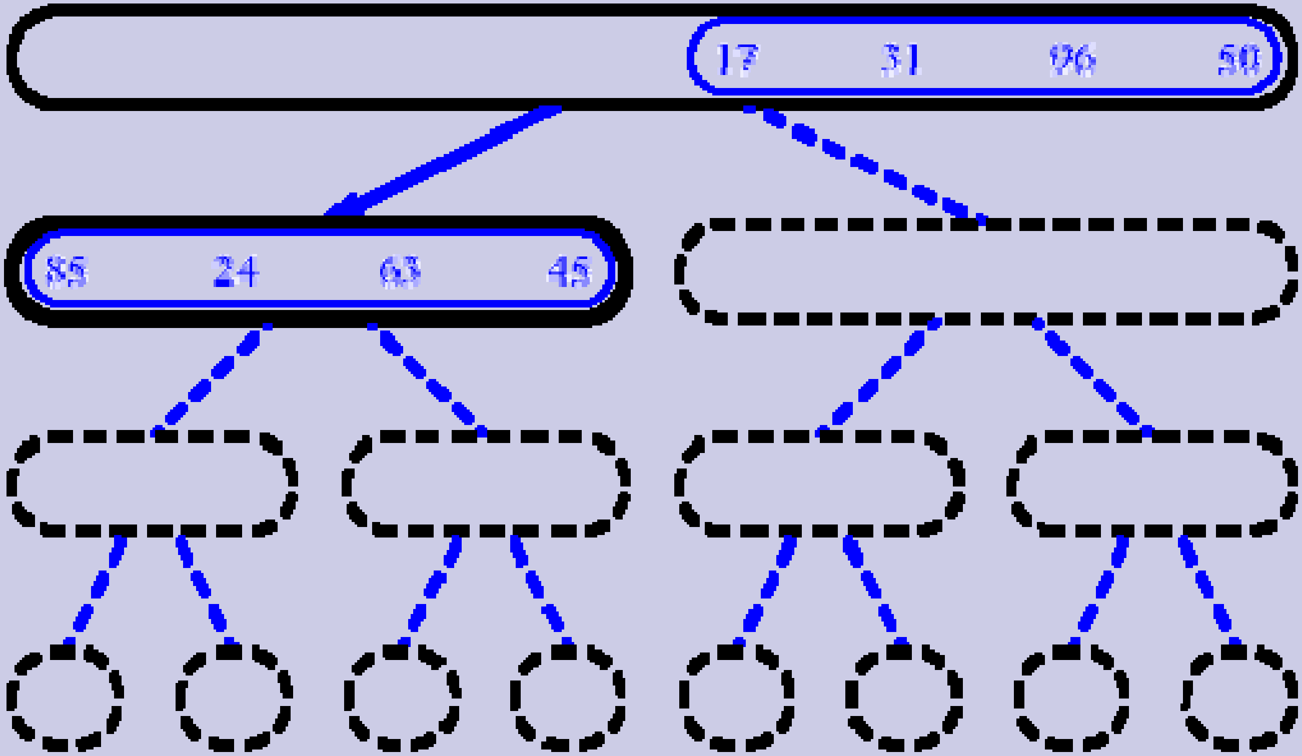

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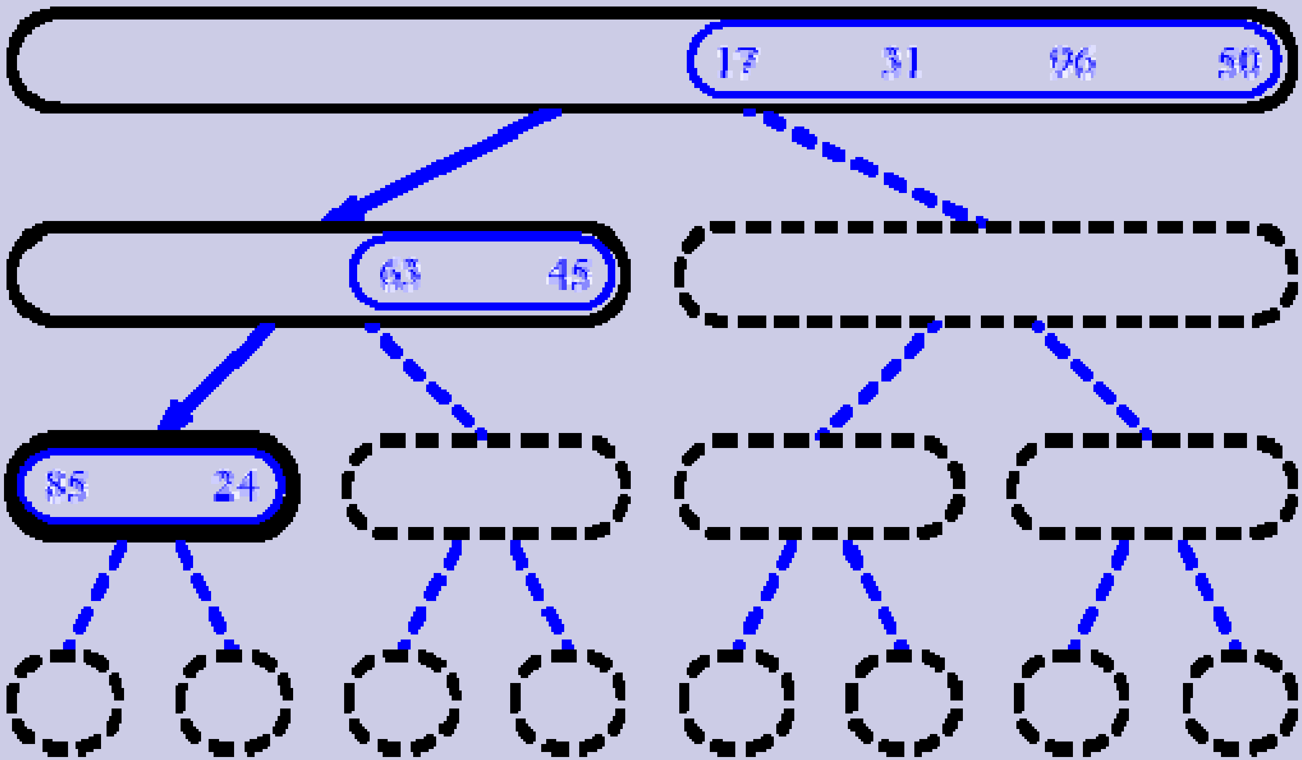

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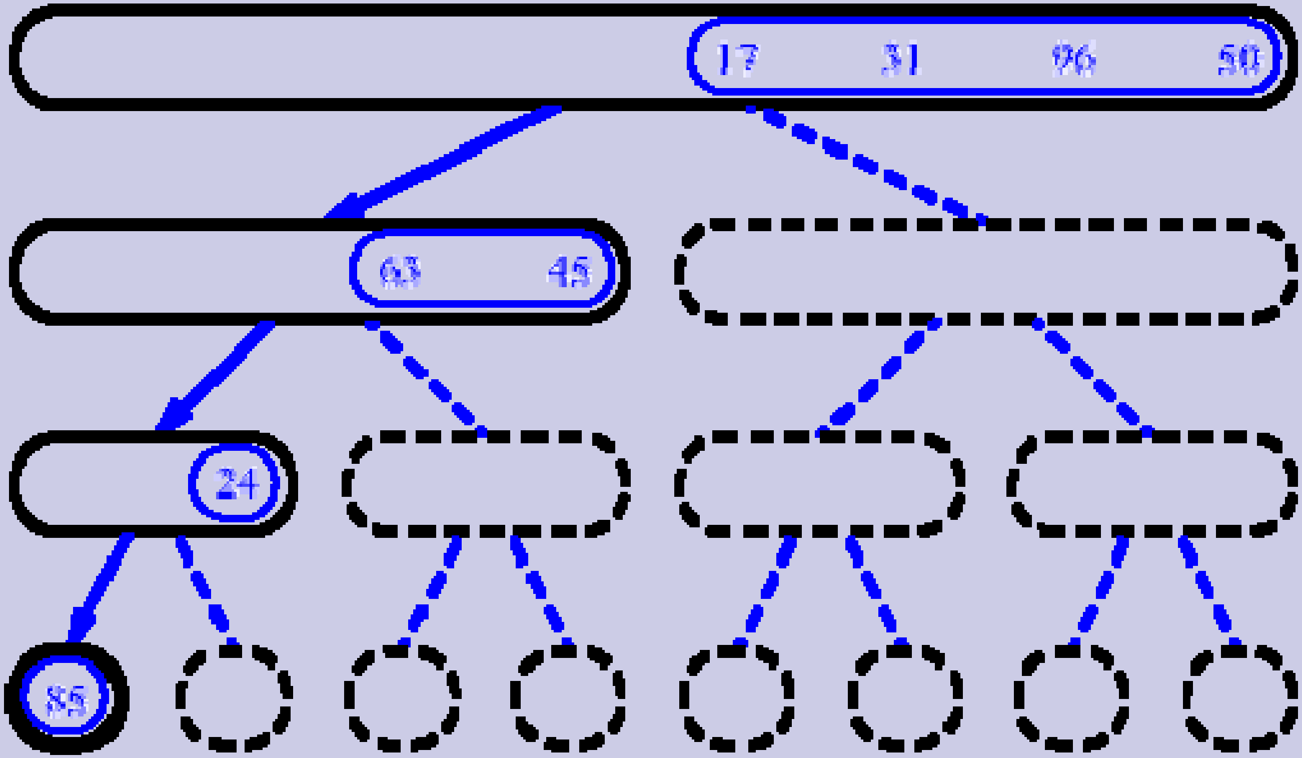

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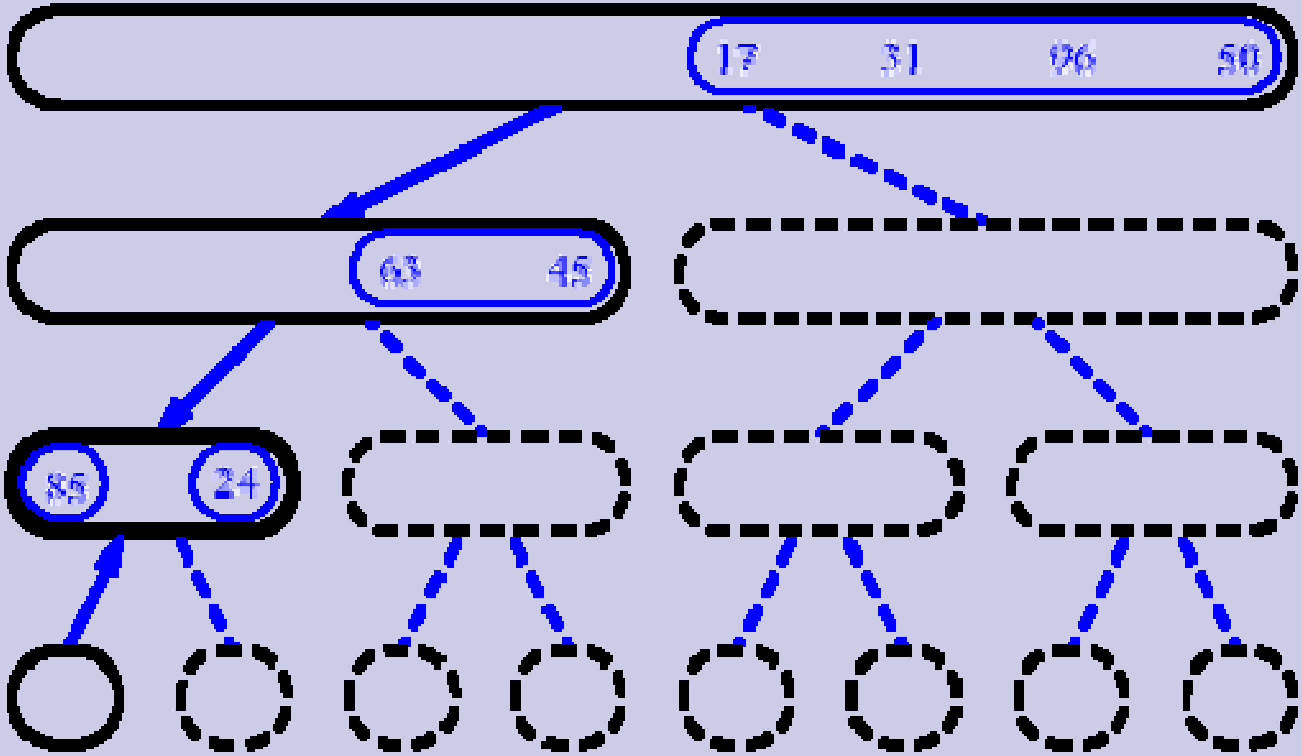

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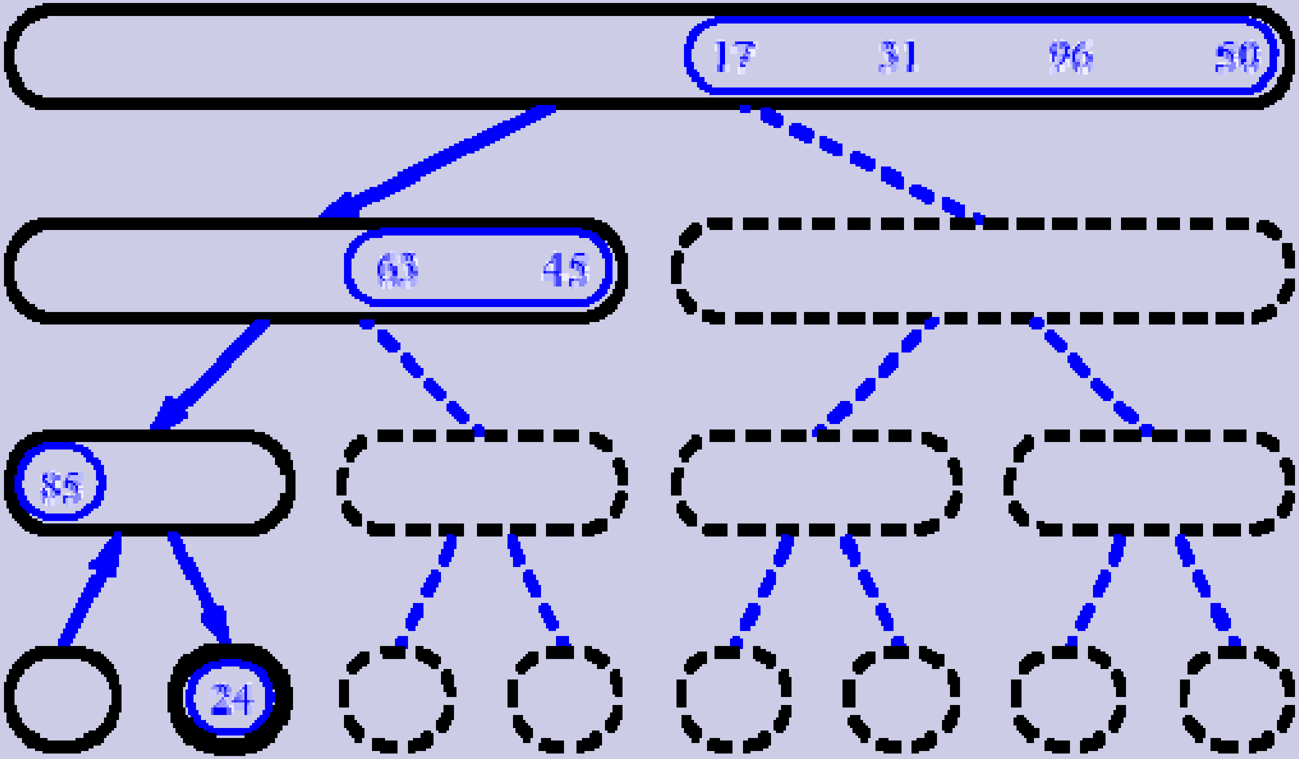

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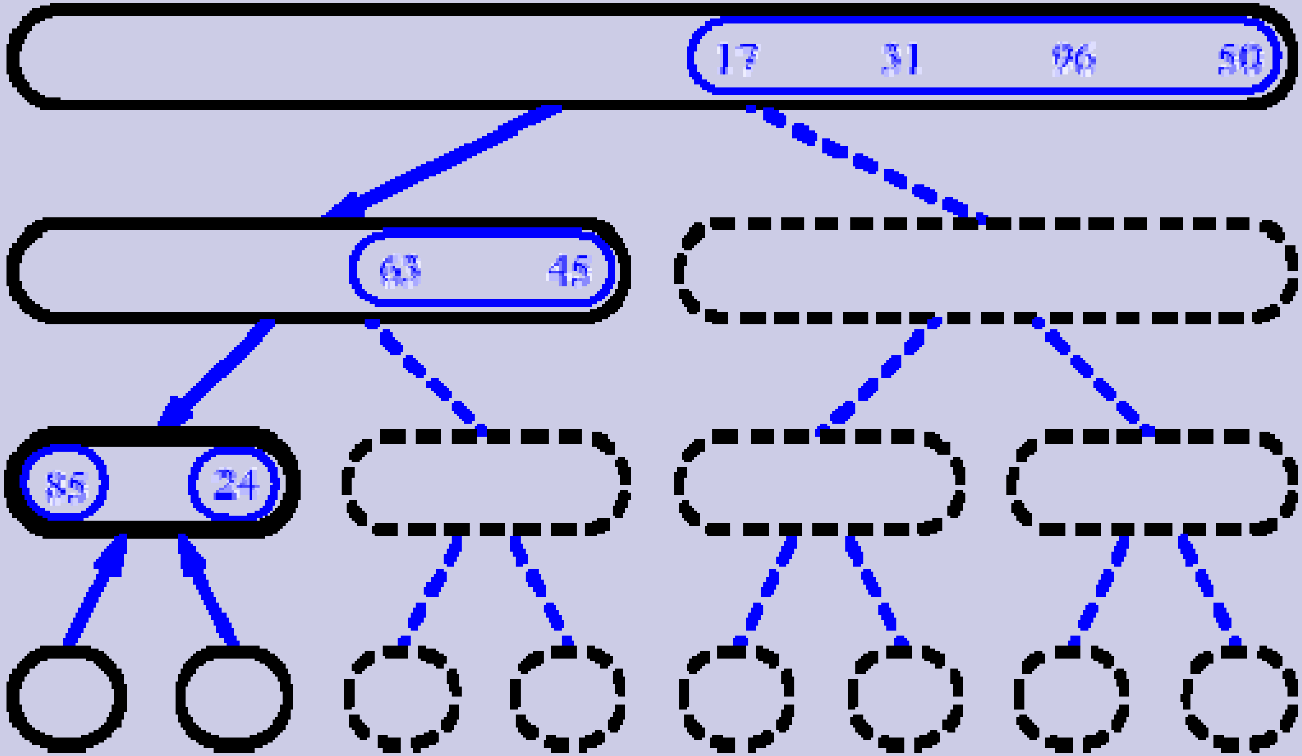

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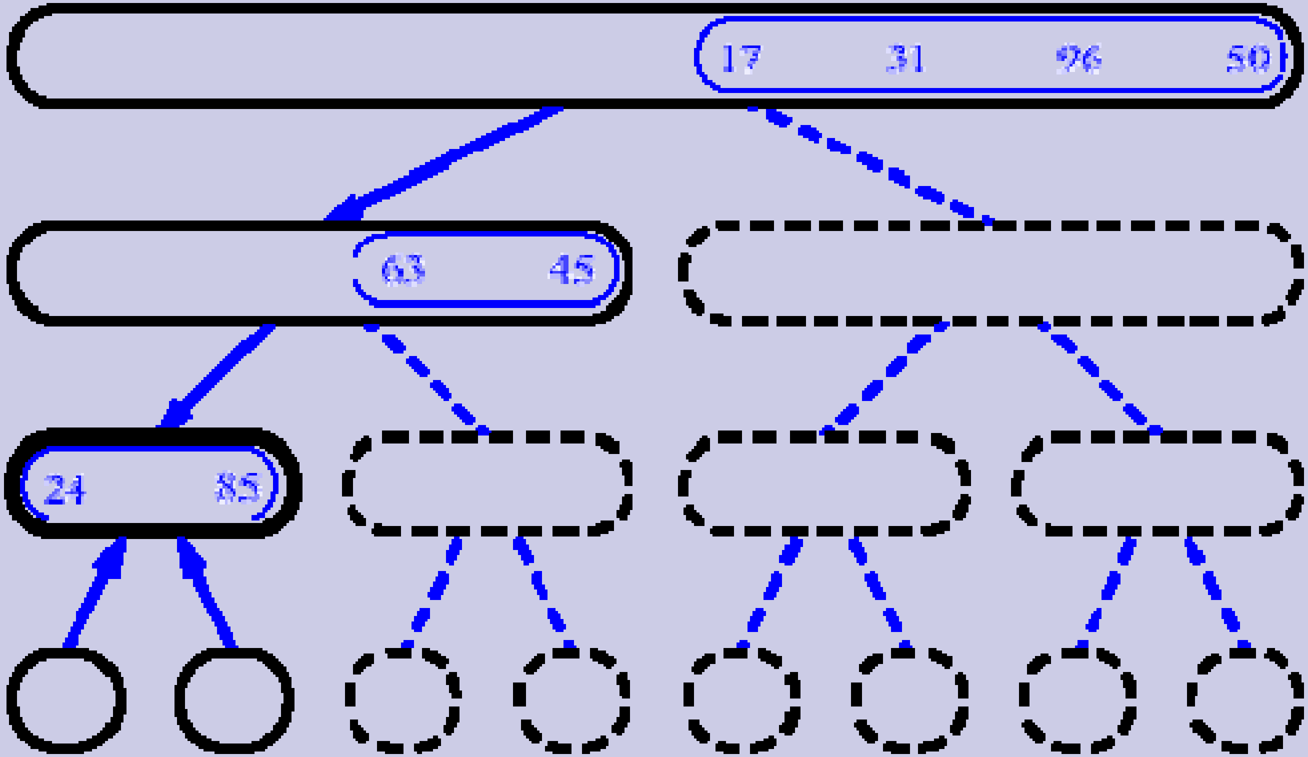

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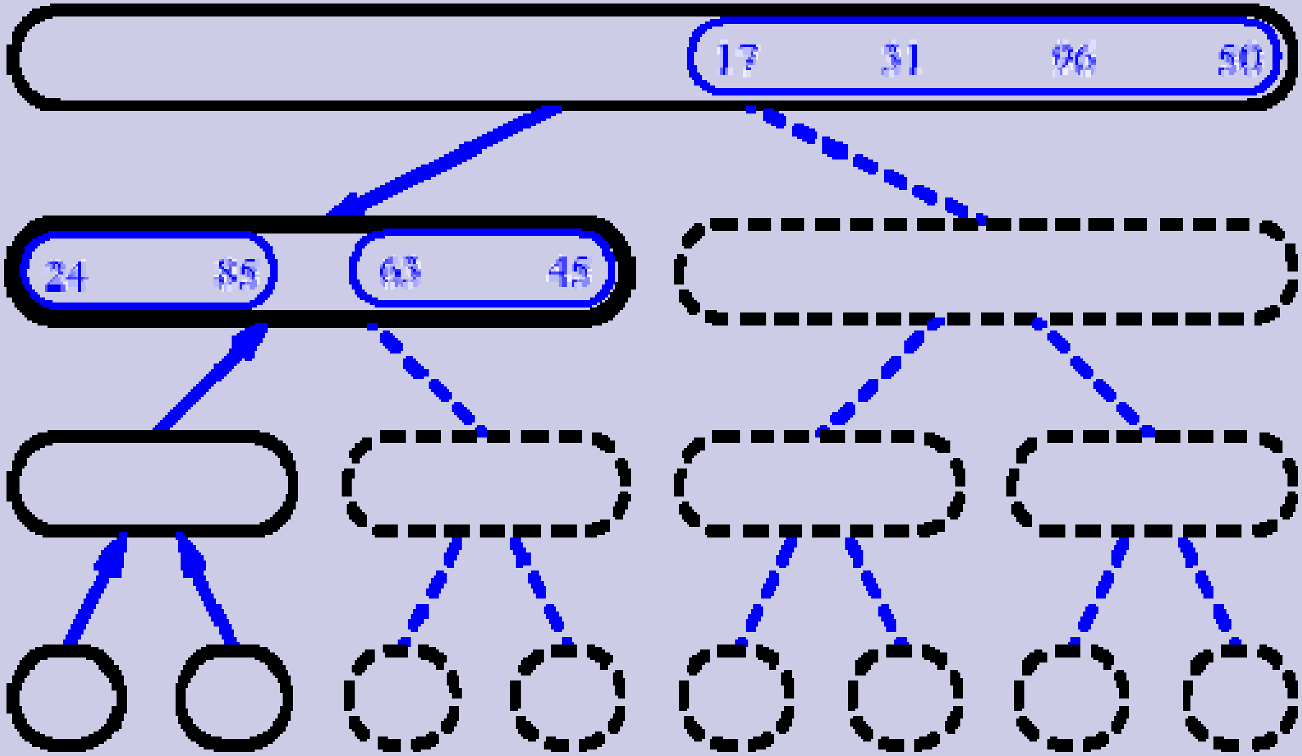

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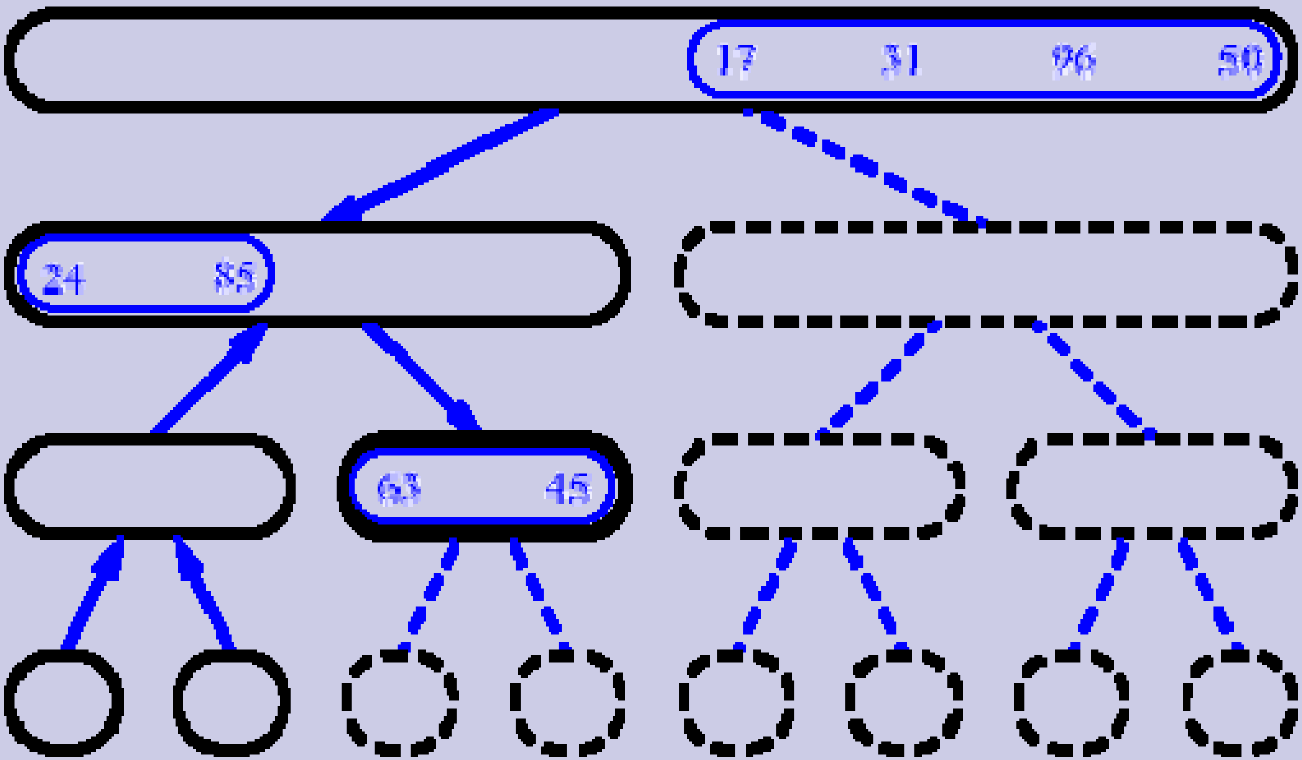

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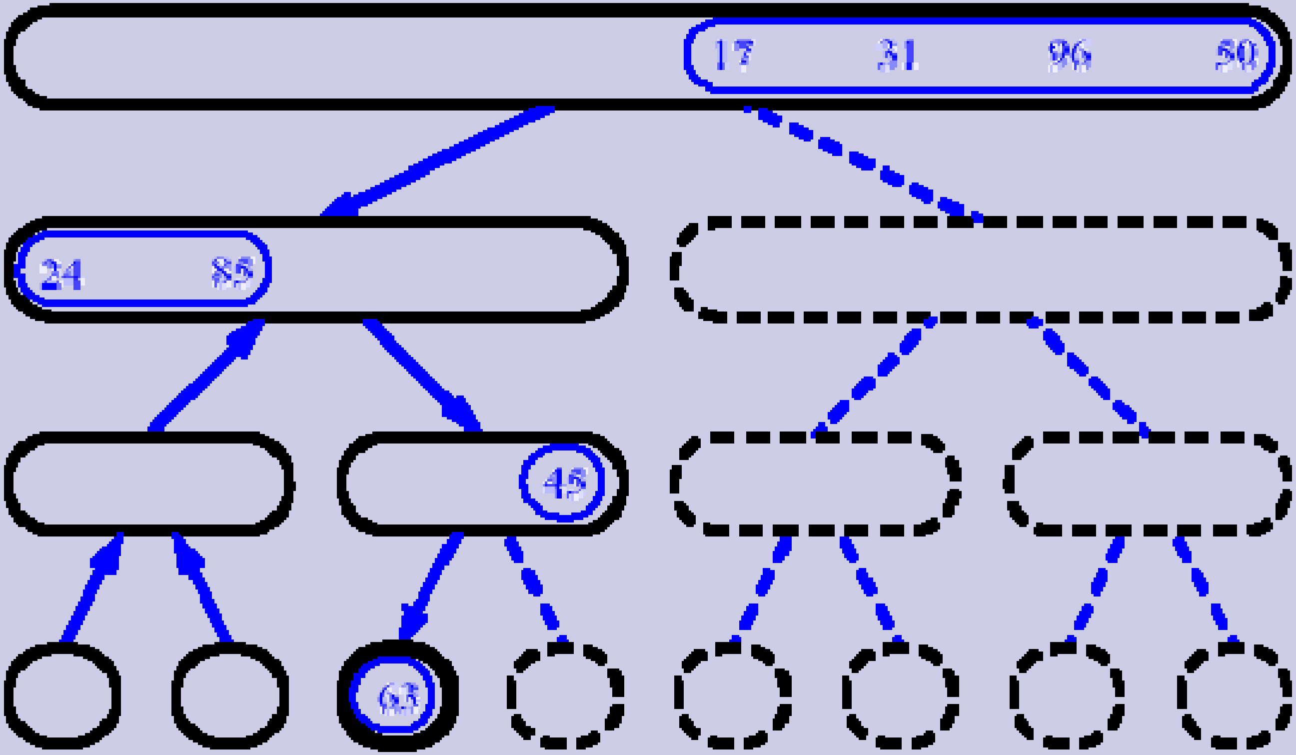

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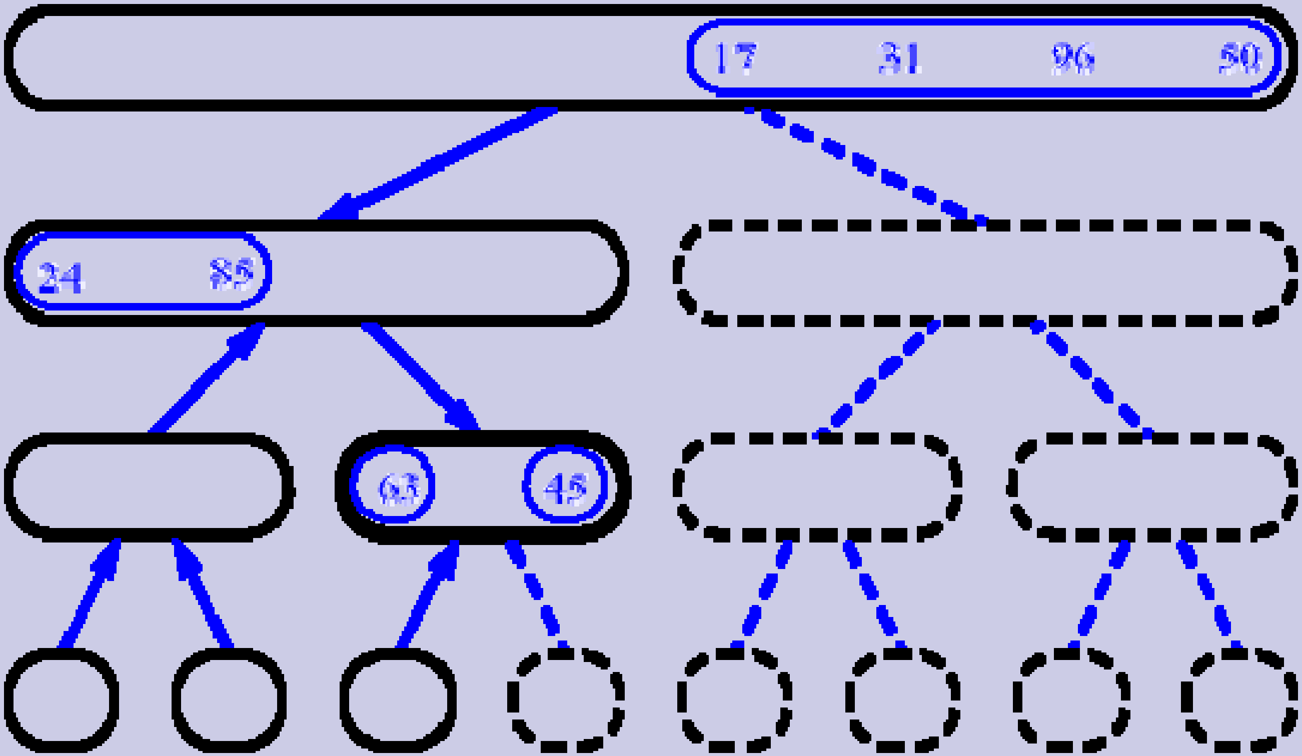

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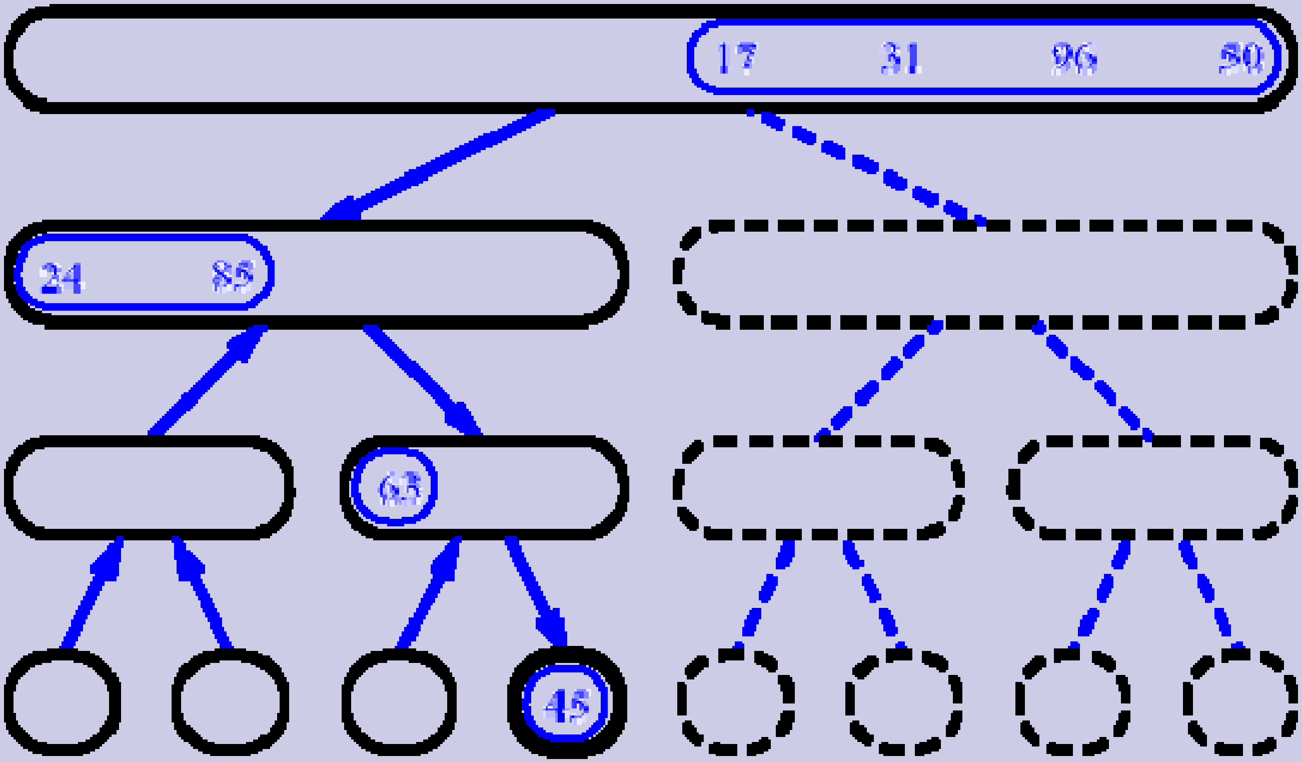

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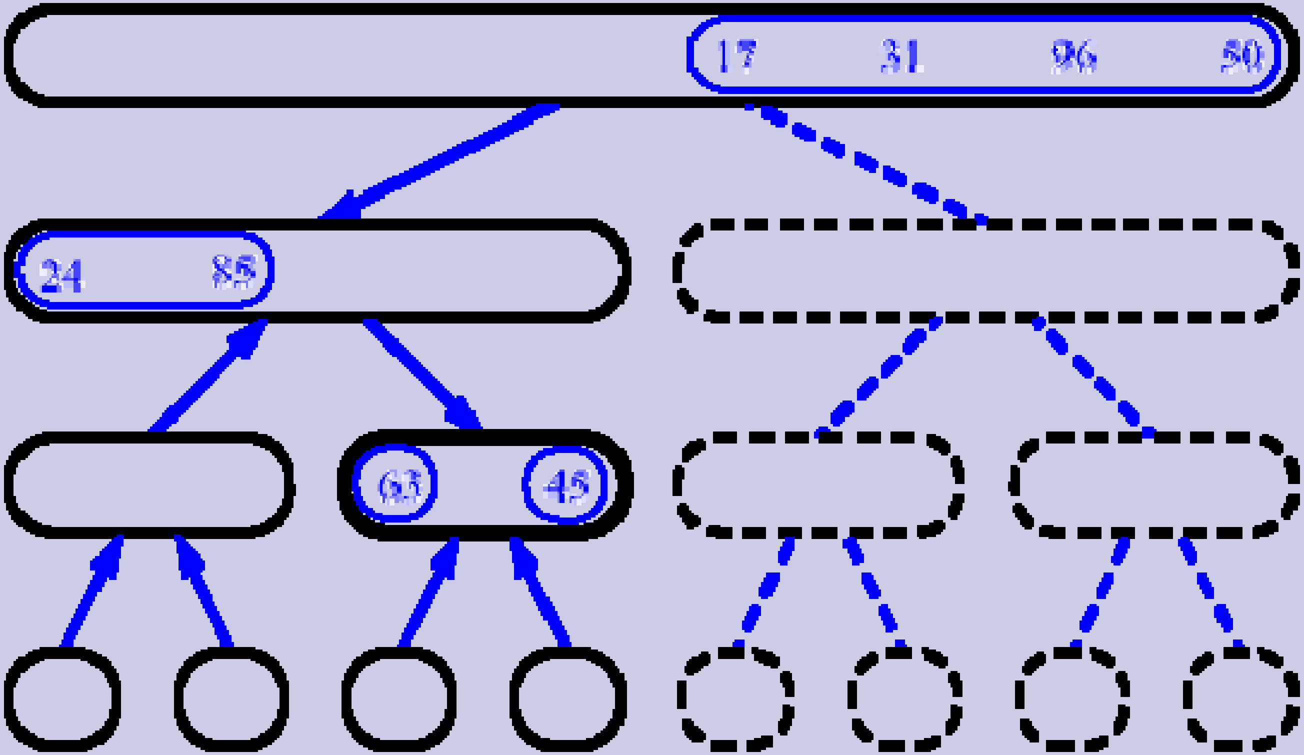

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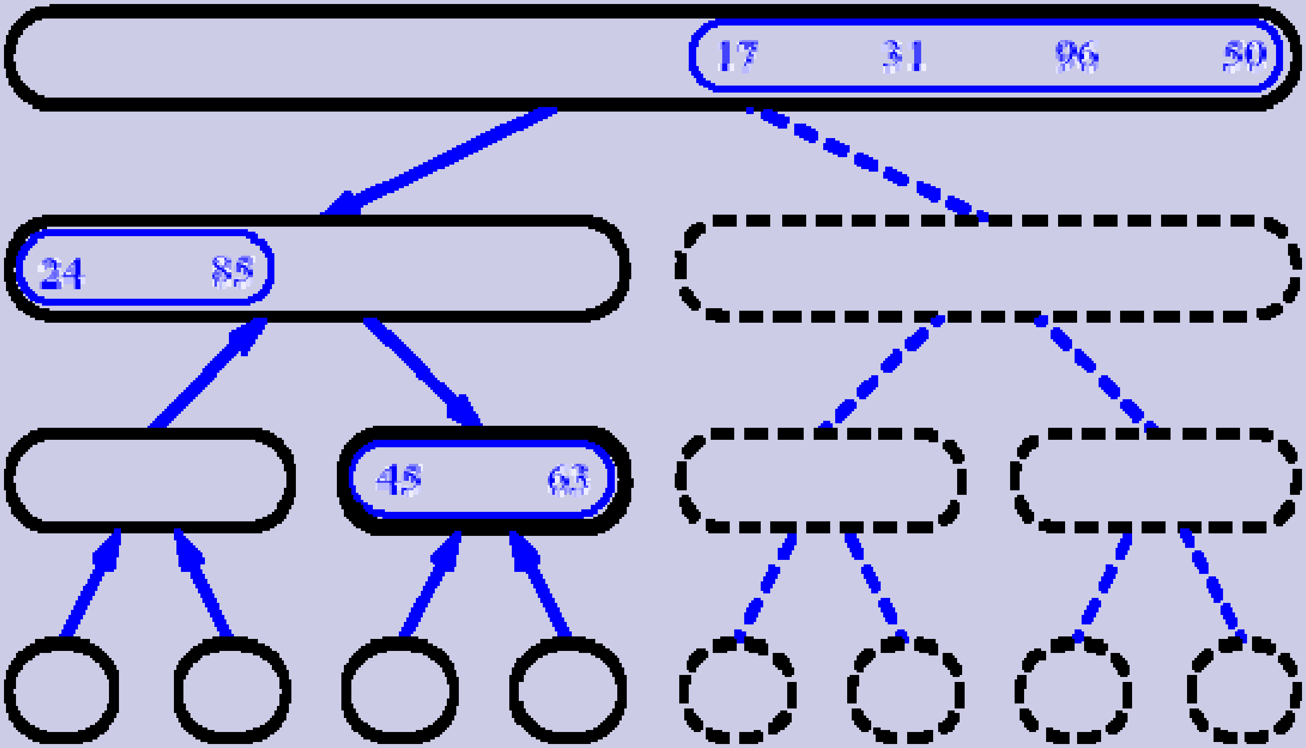

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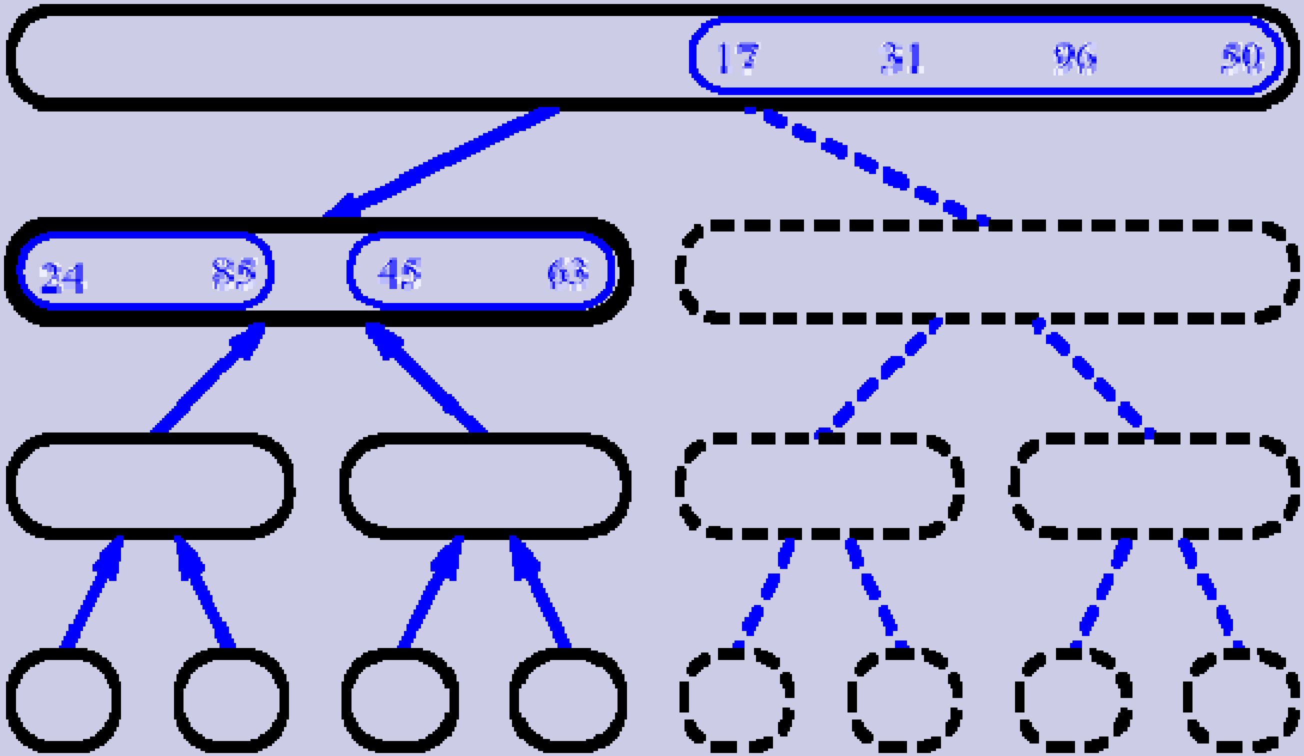

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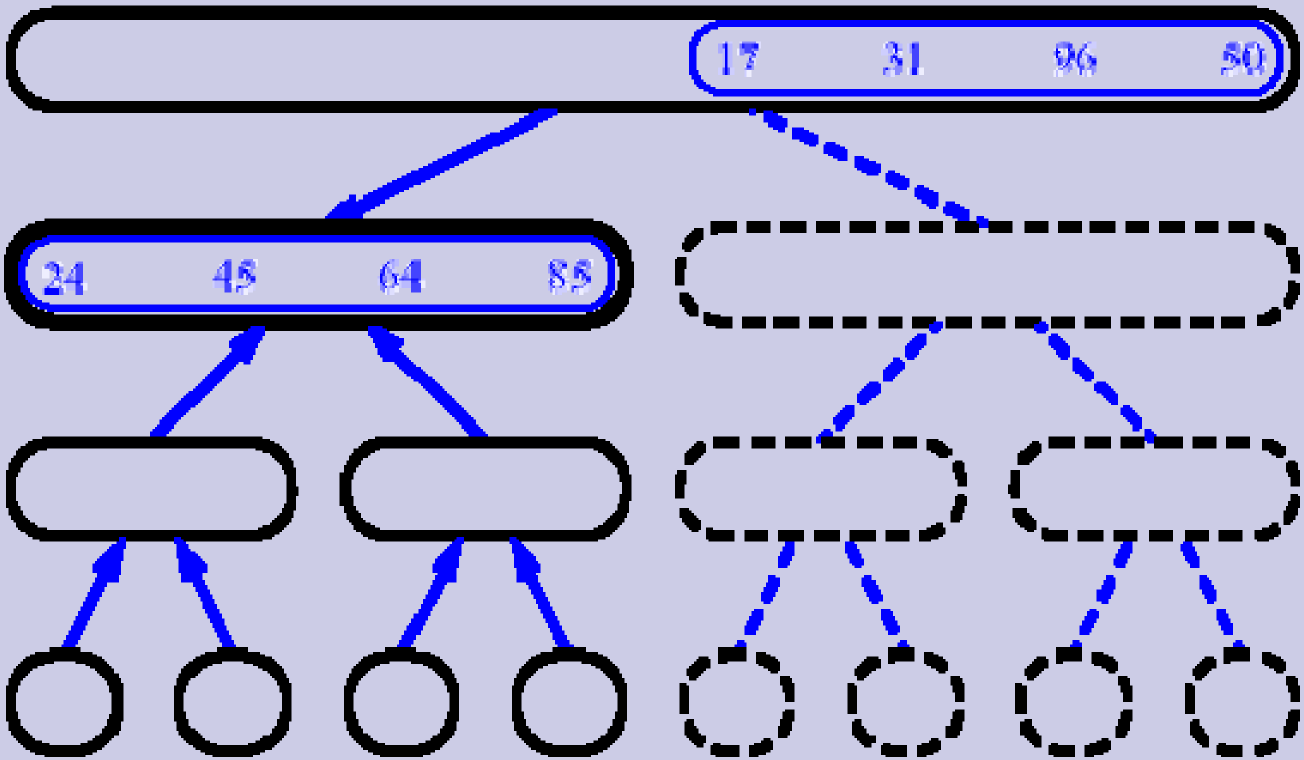

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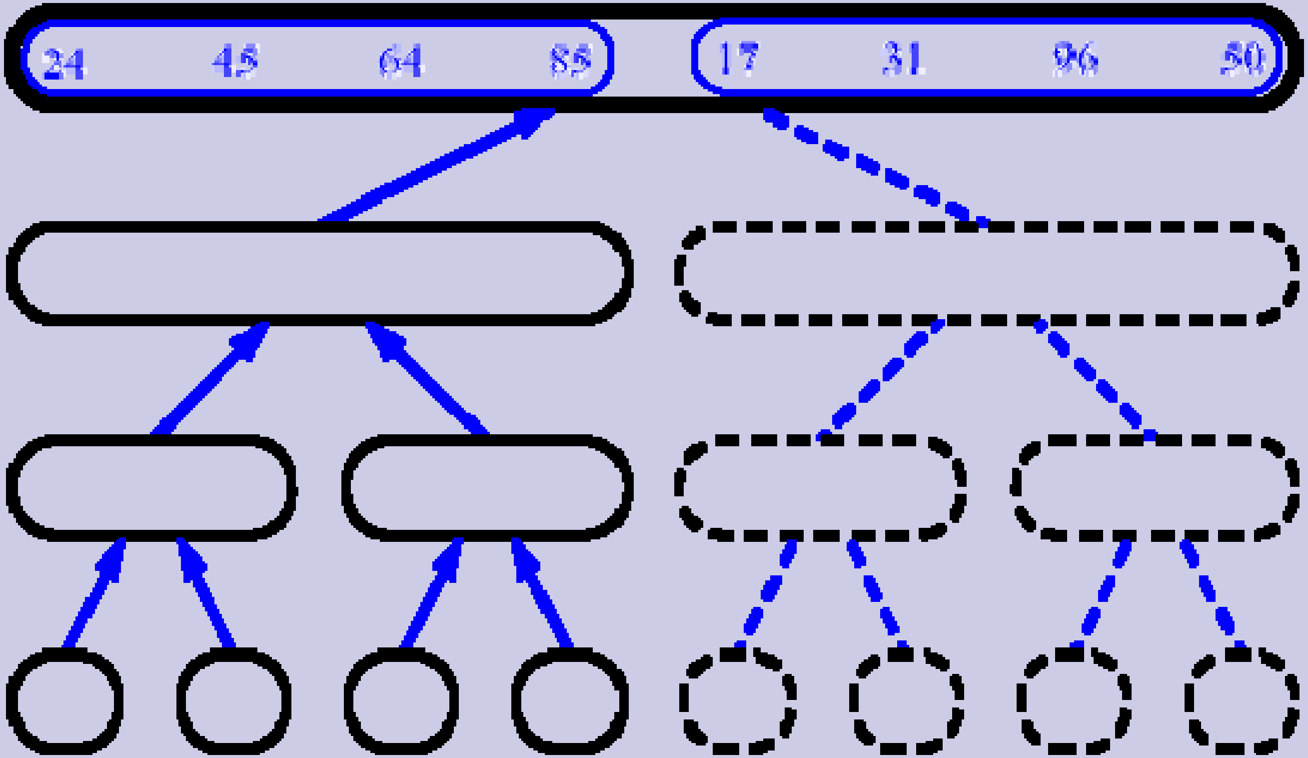

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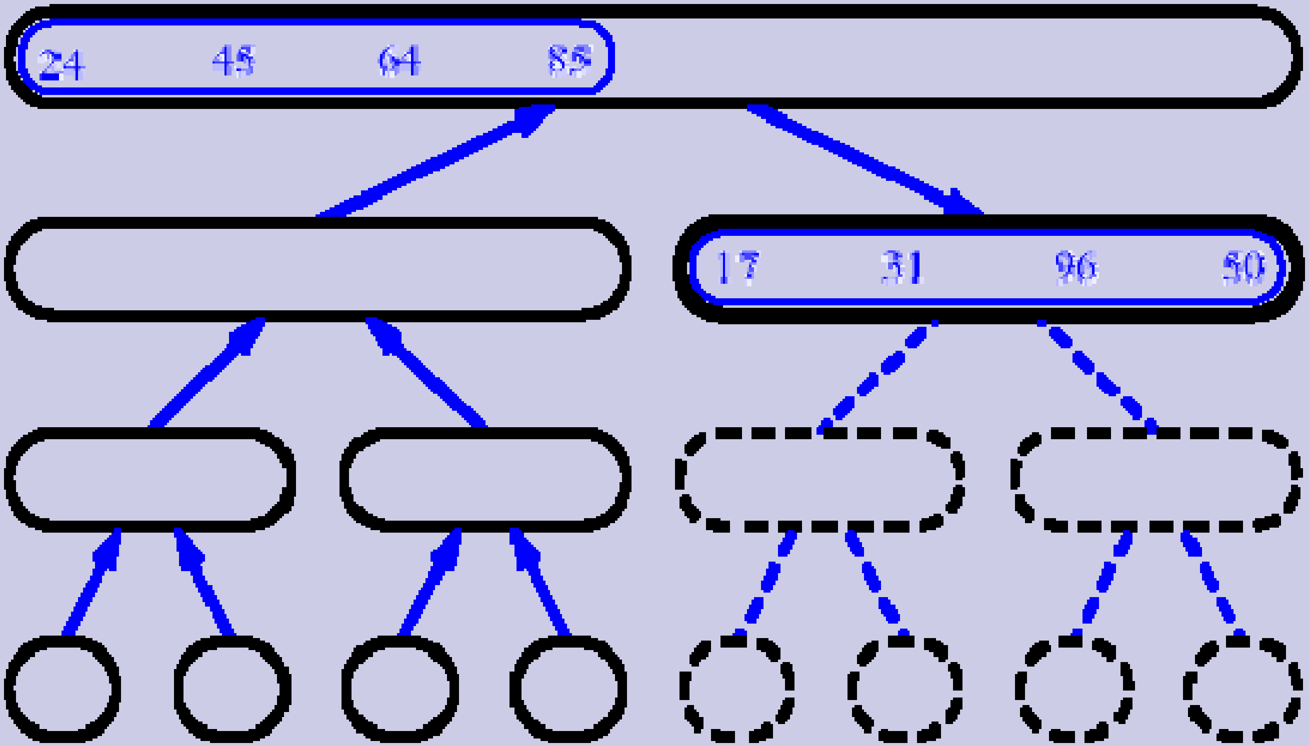

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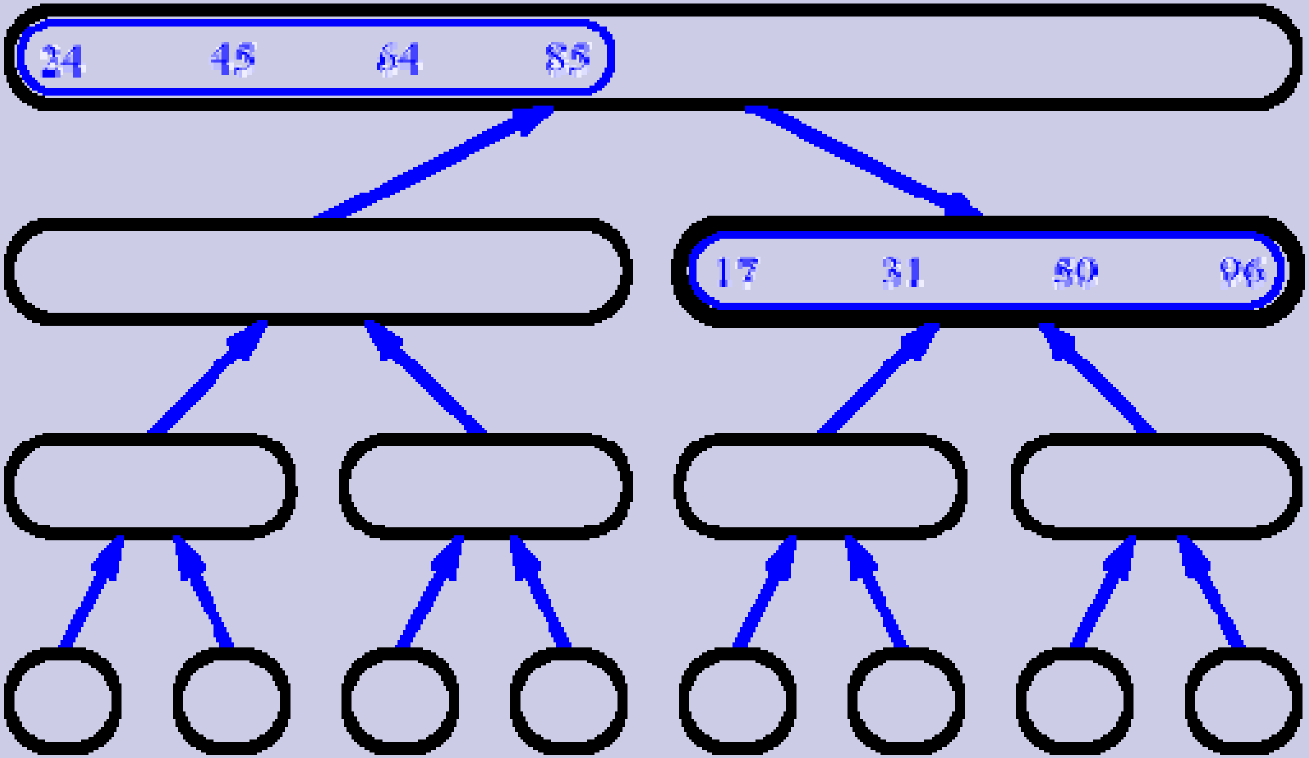

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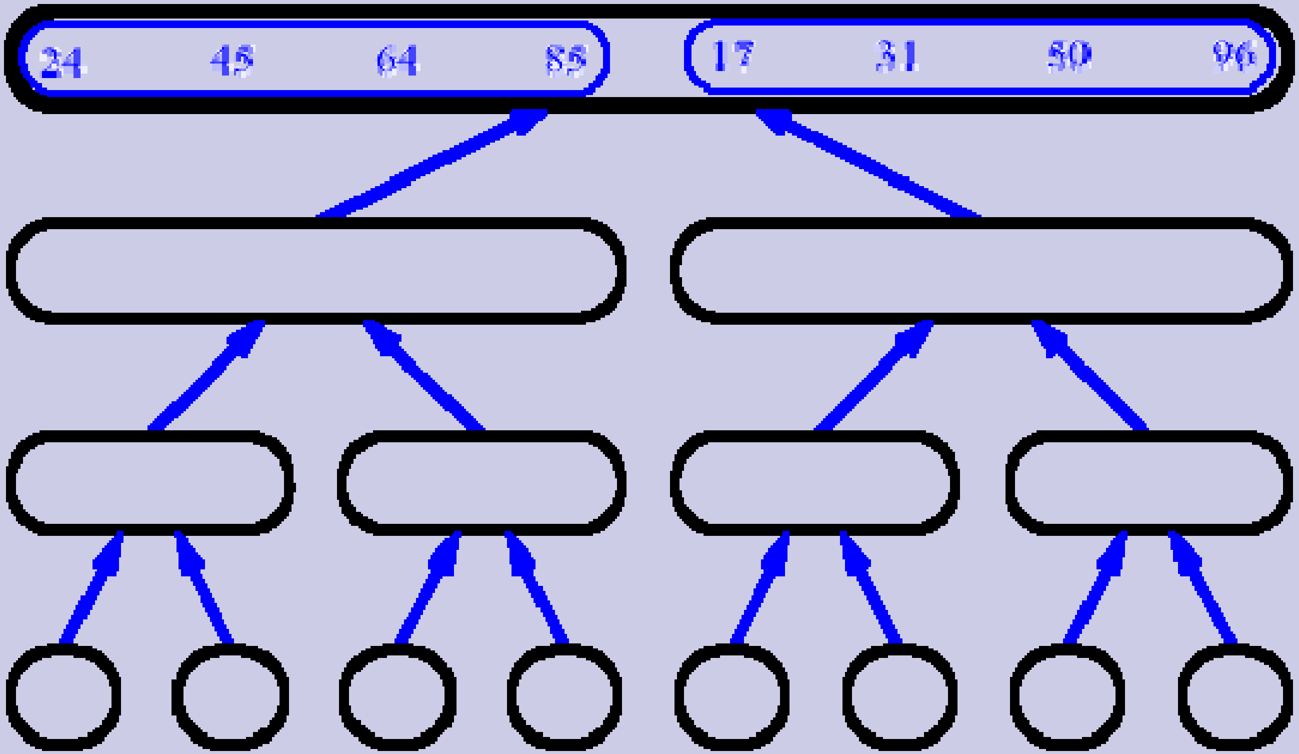

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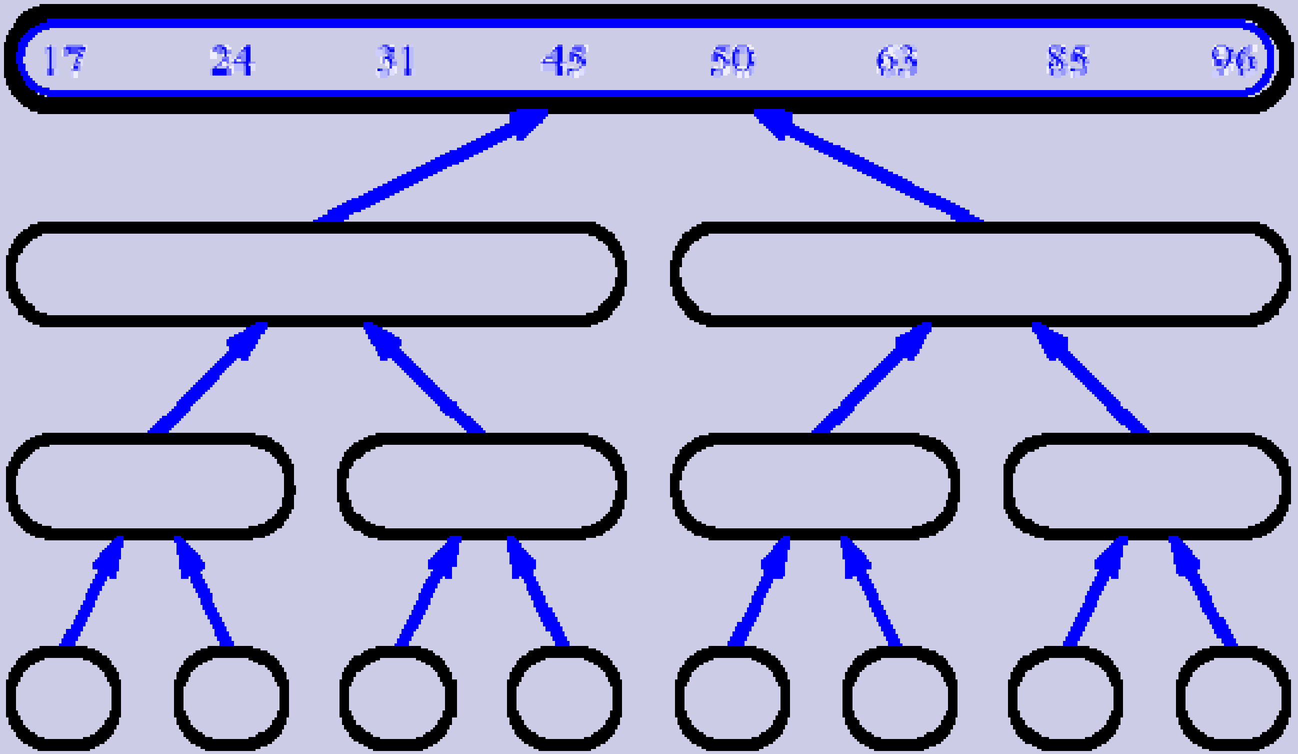
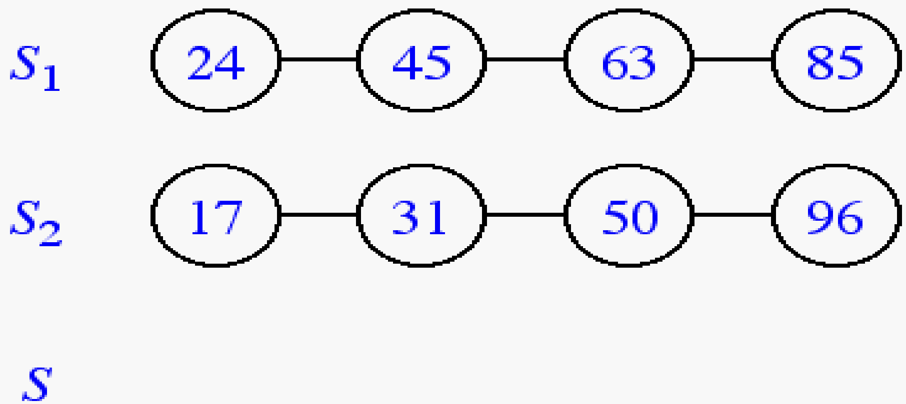


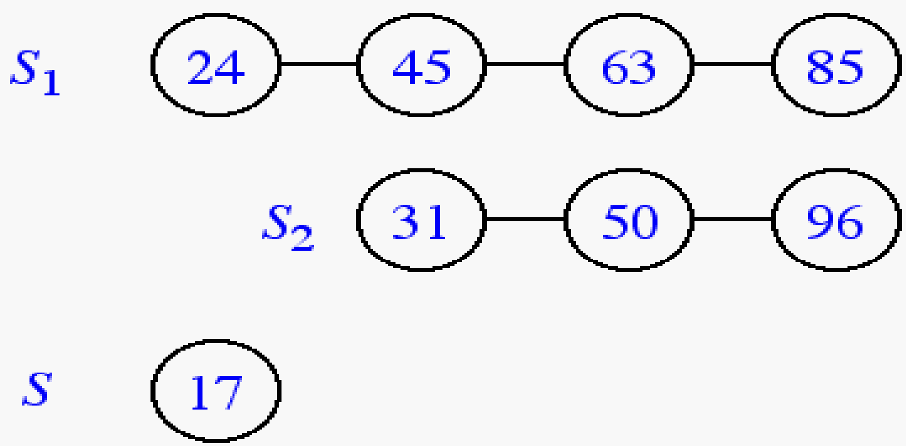
Merging Two Sequences (cont.)

- Some pictures:

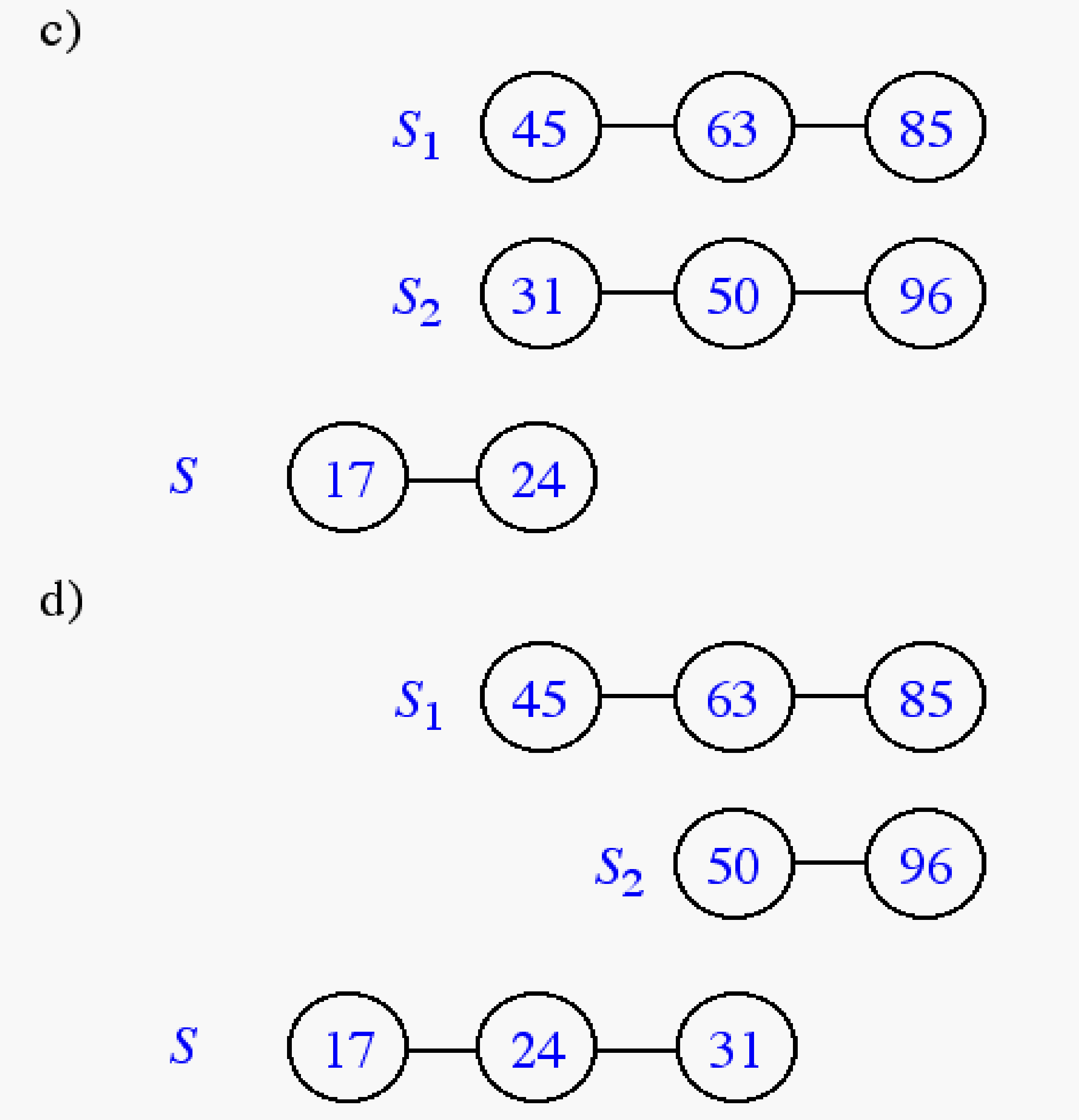
a)



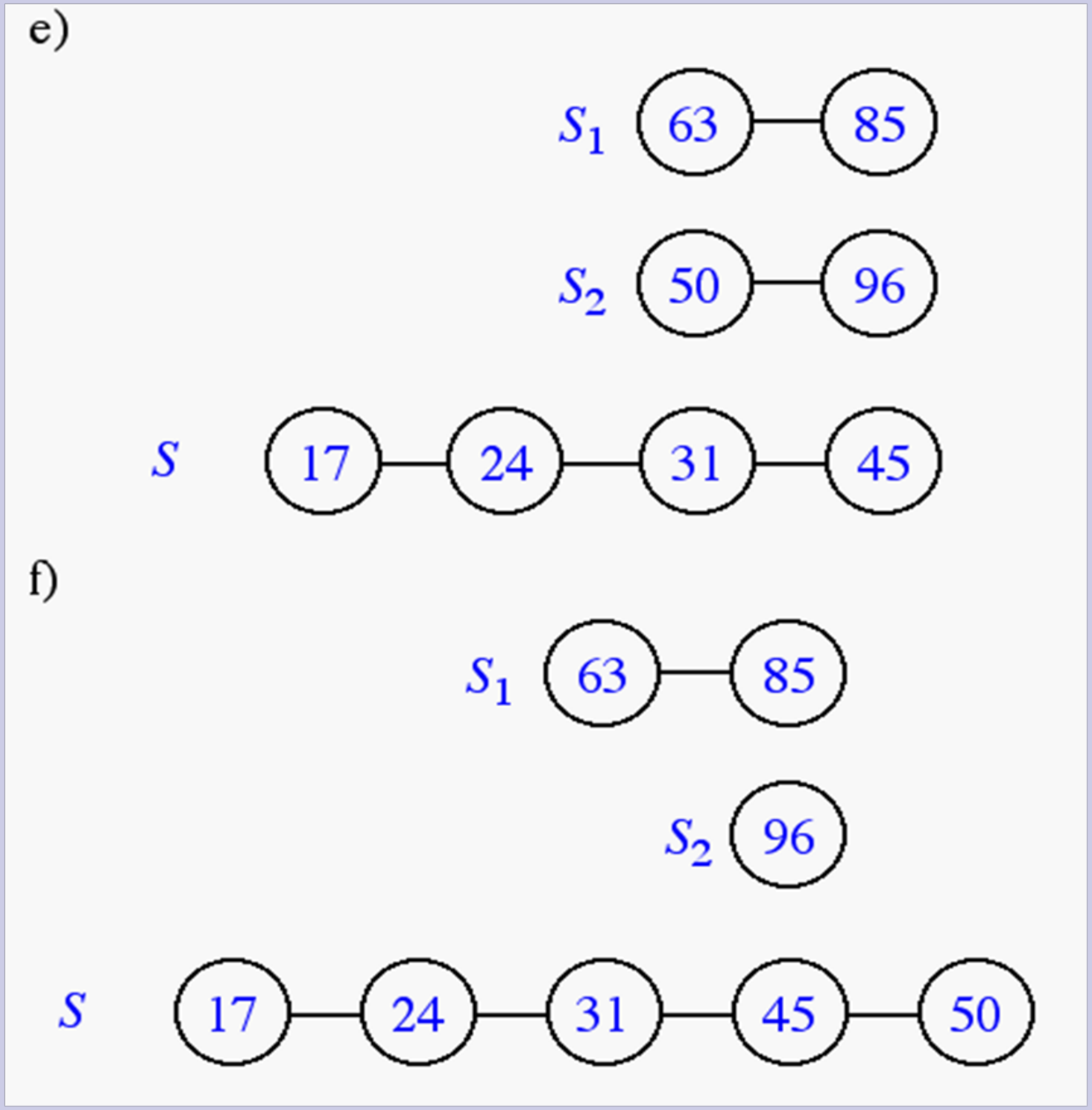
b)



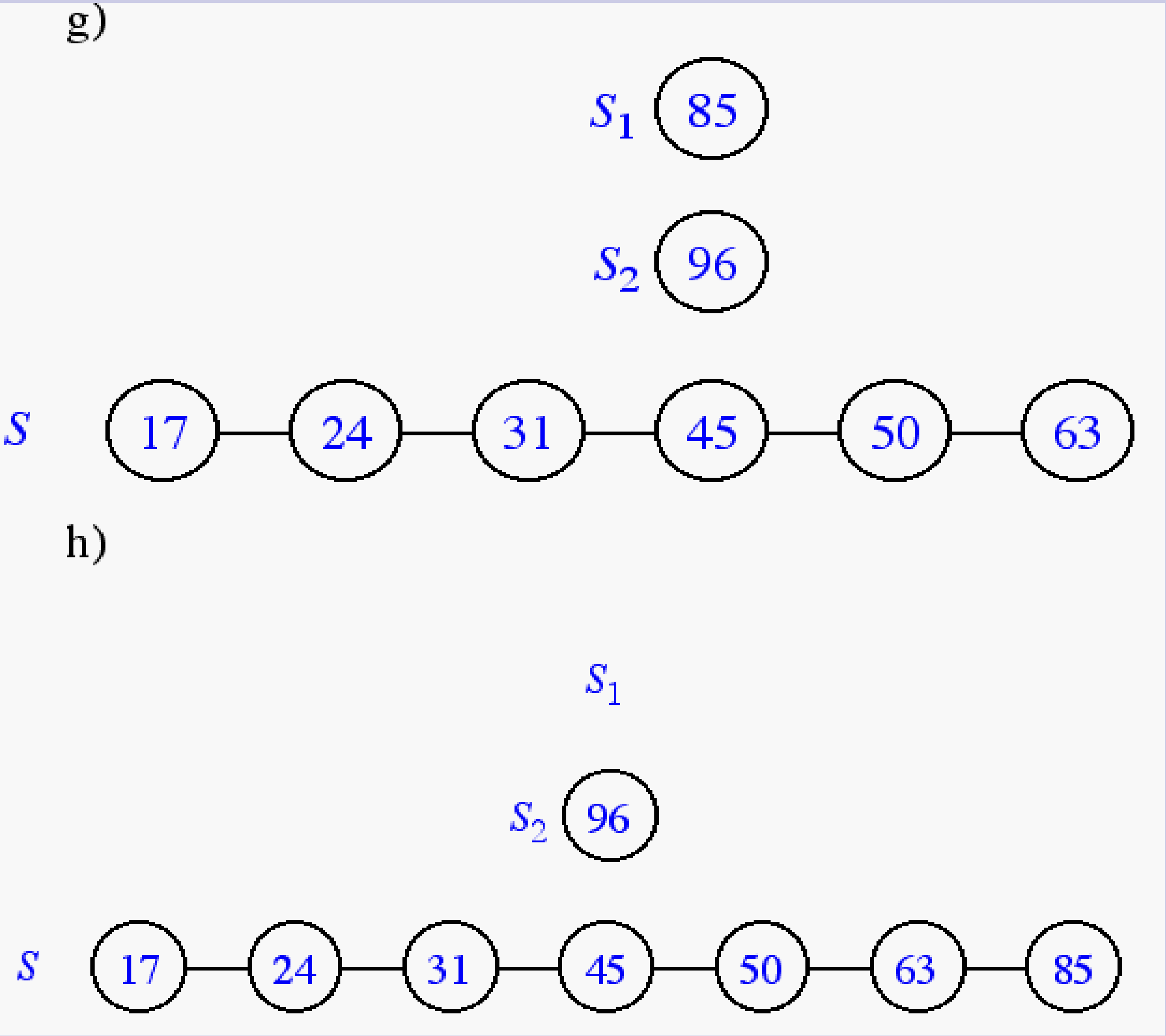
Merging Two Sequences (cont.)



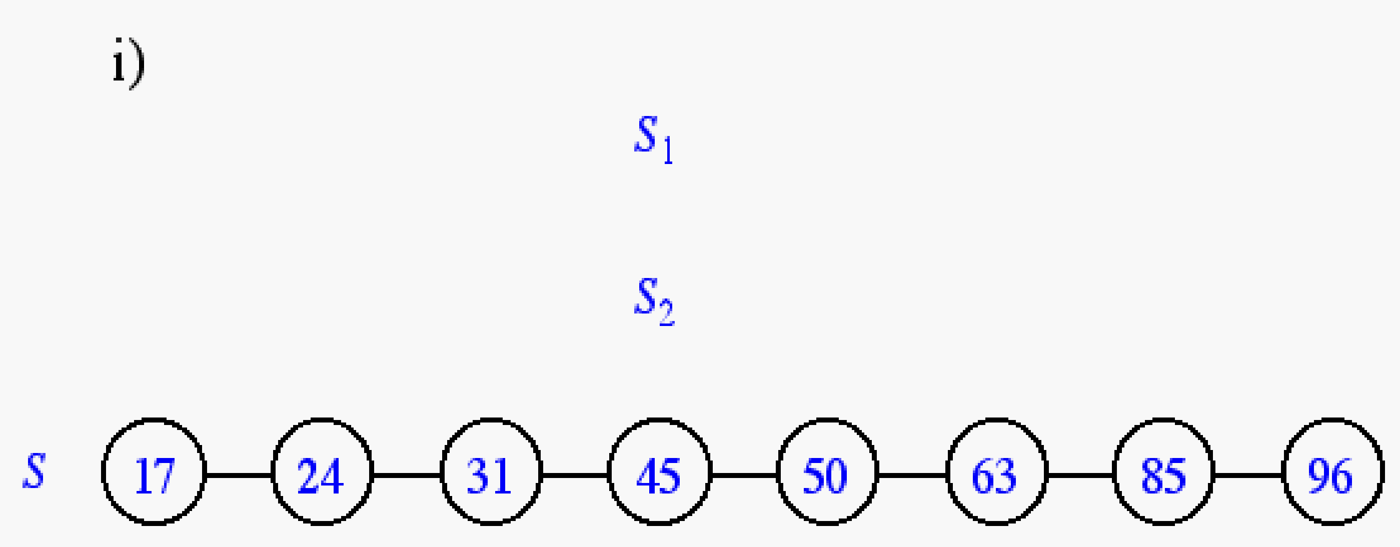
Merging Two Sequences (cont.)



Merging Two Sequences (cont.)

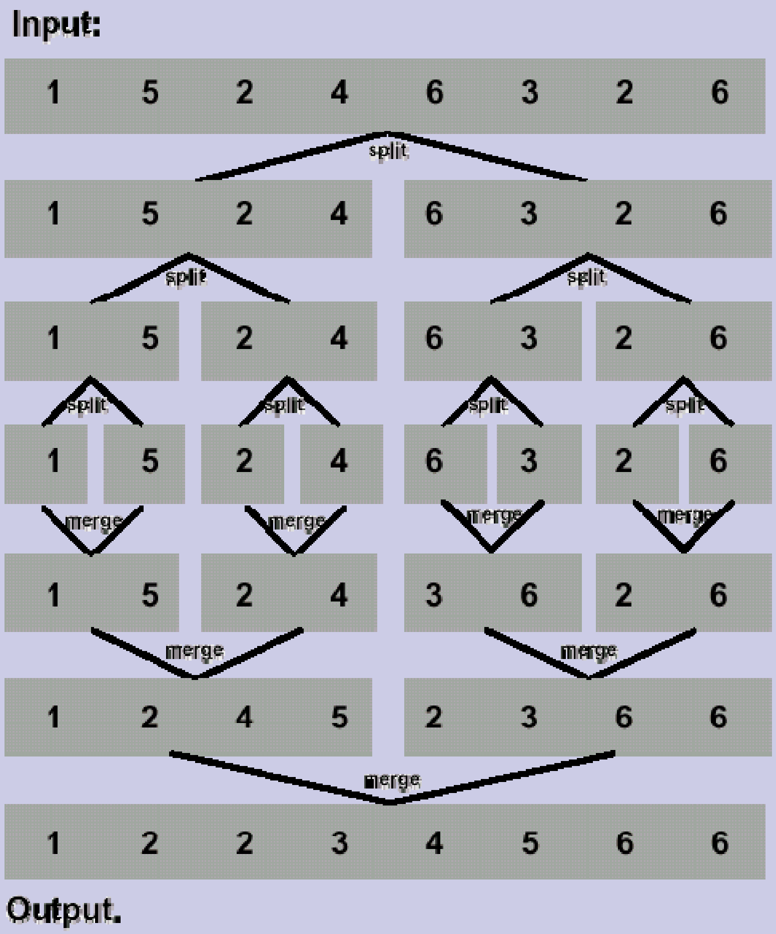


Merging Two Sequences (cont.)



Merge Sort Revisited

- To sort n numbers
  - if n=1 done!
  - recursively sort 2 lists of numbers $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ elements
  - merge 2 sorted lists in $\Theta(n)$ time
- Strategy
  - break problem into similar (smaller) subproblems
  - recursively solve subproblems
  - combine solutions to answer



Recurrences

- Running times of algorithms with **Recursive calls** can be described using recurrences
- A **recurrence** is an equation or inequality that describes a function in terms of its value on smaller inputs

$$T(n) = \begin{cases} \text{solving\_trivial\_problem} & \text{if } n = 1 \\ \text{num\_pieces } T(n / \text{subproblem\_size\_factor}) + \text{dividing} + \text{combining} & \text{if } n > 1 \end{cases}$$

- Example: Merge Sort

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Solving Recurrences

- Repeated substitution method
  - Expanding the recurrence by substitution and noticing patterns
- Substitution method
  - guessing the solutions
  - verifying the solution by the mathematical induction
- Recursion-trees
- Master method
  - templates for different classes of recurrences

Repeated Substitution Method

- Let's find the running time of merge sort (let's assume that $n=2^b$, for some b).

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n > 1 \end{cases}$$

$$\begin{aligned}
T(n) &= 2T\left(n/2\right) + n && \text{substitute} \\
&= 2\left(2T\left(n/4\right) + n/2\right) + n && \text{expand} \\
&= 2^2 T(n/4) + 2n && \text{substitute} \\
&= 2^2 (2T(n/8) + n/4) + 2n && \text{expand} \\
&= 2^3 T(n/8) + 3n && \text{observe the pattern} \\
T(n) &= 2^i T(n/2^i) + in \\
&= 2^{\lg n} T(n/n) + n \lg n = n + n \lg n
\end{aligned}$$

Repeated Substitution Method

- The procedure is straightforward:
  - Substitute
  - Expand
  - Substitute
  - Expand
  - …
  - Observe a pattern and write how your expression looks after the i-th substitution
  - Find out what the value of i (e.g., $\lg n$) should be to get the base case of the recurrence (say $T(1)$)
  - Insert the value of $T(1)$ and the expression of i into your expression

Java Implementation of Merge-Sort

```
public interface SortObject {
  //sort sequence S in nondecreasing order
  using compartor c
  public void sort (Sequence S, Comparator c);
}
```

Java Implementation of MergeSort (cont.)

```
public class ListMergeSort implements SortObject {

public void sort(Sequence S, Comparator c) {
   int n = S.size();
   if (n < 2) return; //sequence with 0/1 element is sorted.
   // divide
   Sequence S1 = (Sequence)S.newContainer();
   // put the first half of S into S1
   for (int i=1; i <= (n+1)/2; i++) {
      S1.insertLast(S.remove(S.first()));
      }
   Sequence S2 = (Sequence)S.newContainer();
   // put the second half of S into S2
   for (int i=1; i <= n/2; i++) {
      S2.insertLast(S.remove(S.first()));
      }
   sort(S1,c); // recur
   sort(S2,c);
   merge(S1,S2,c,S); // conquer
  }
```

Java Implementation of MergeSort (cont.)

```
public void merge(Sequence S1, Sequence S2, Comparator c, Sequence S) {
        while(!S1.isEmpty() && !S2.isEmpty()) {
                if(c.isLessThanOrEqualTo(S1.first().element(),
                        S2.first().element())) {
                        // S1's 1st elt <= S2's 1st elt
                        S.insertLast(S1.remove(S1.first()));
                }else { // S2's 1st elt is the smaller one
                        S.insertLast(S2.remove(S2.first()));
                }
        }if(S1.isEmpty()) {
                while(!S2.isEmpty()) {
                        S.insertLast(S2.remove(S2.first()));
                }
        }if(S2.isEmpty()) {
                while(!S1.isEmpty()) {
                        S.insertLast(S1.remove(S1.first()));
                }}}
```