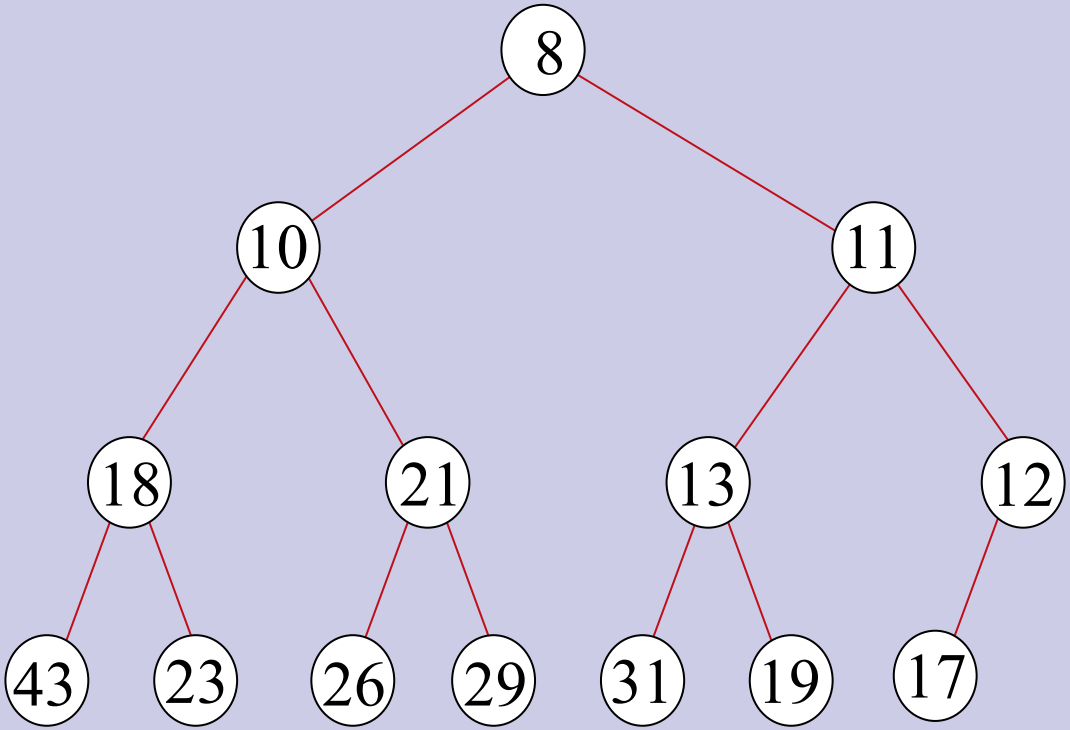
Binary Heaps

- Delete-min
- Building a heap in O(n) time
- Heap Sort

Delete-min

- The minimum element is the one at the top of the heap.
- We can delete this and move one of its children up to fill the space.
- Empty location moves down the tree.
- Might end up at any position on last level.
- Resulting tree would not be left filled.

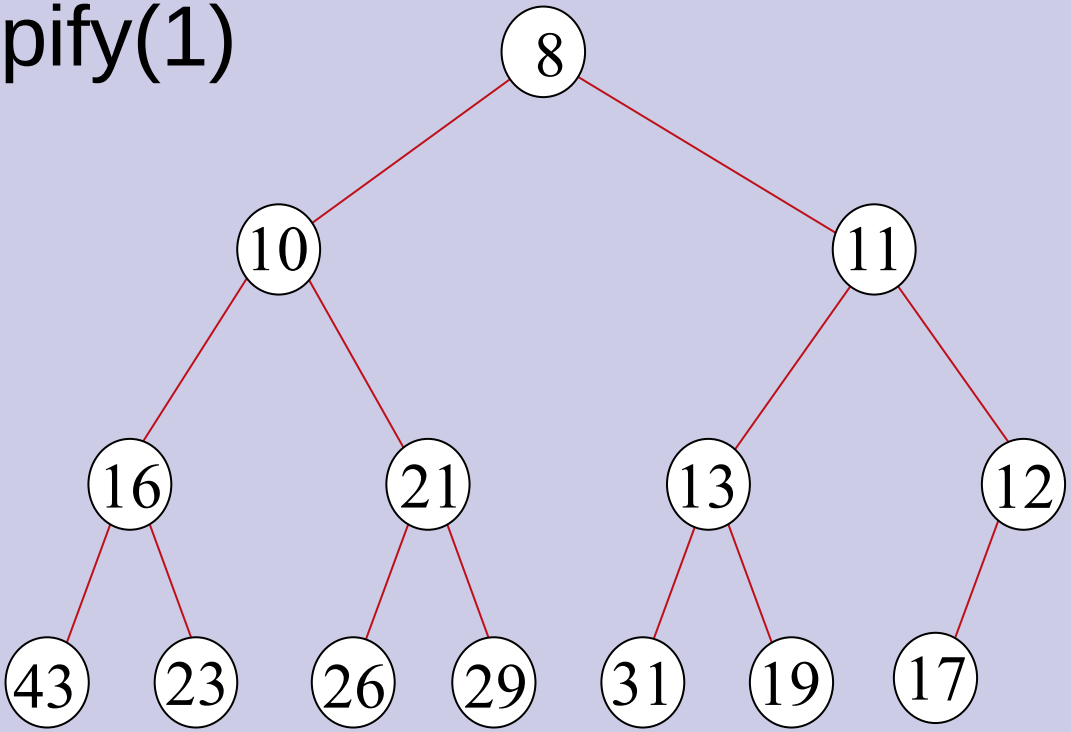
Delete-min in a Heap



This is not a heap

Delete-min in a Heap (2)

- Replace top element with last element of heap.
- Heapify(1)



Building a heap

- We start from the bottom and move up
- All leaves are heaps to begin with

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BUILD-HEAP(A)
 1 for i ← ⌊n/2⌋ downto 1
 2       do HEAPIFY(A, i)
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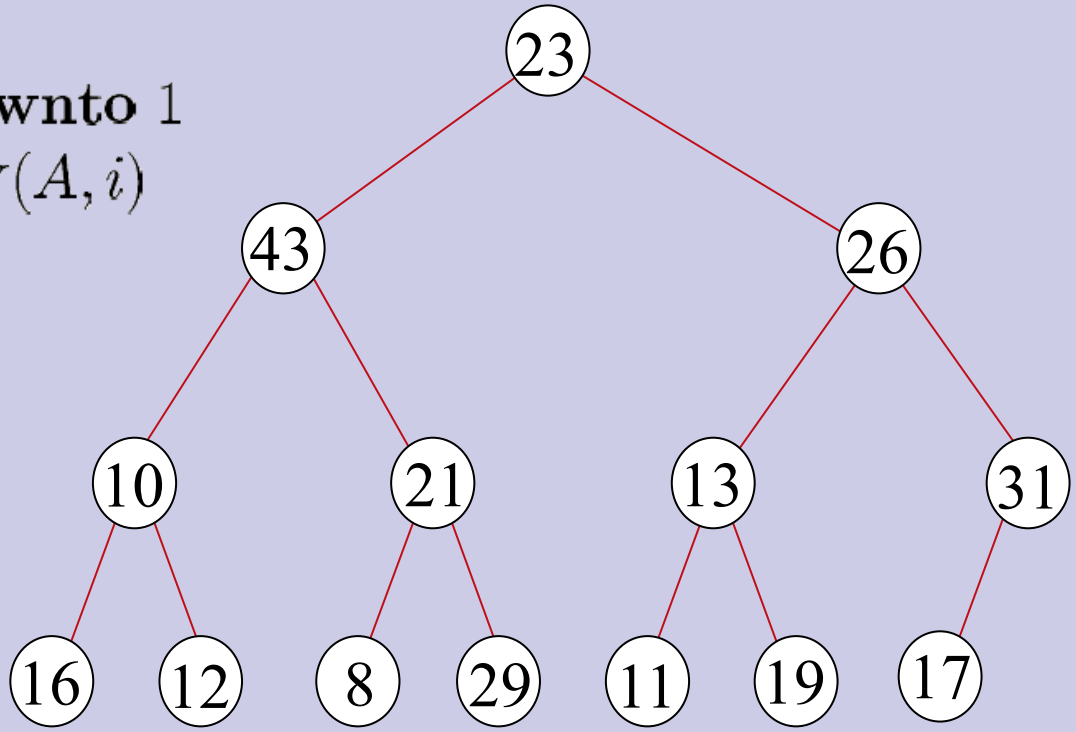


Building a Heap: Analysis

- Correctness: induction on *i*, all trees rooted at *m* > *i* are heaps
- Running time: *n* calls to Heapify = *n O(lg n) = O(n lg n)*
- We can provide a better O(n) bound.
  - Intuition: for most of the time Heapify works on smaller than *n* element heaps

Building a Heap: Analysis (2)

- height of node: length of longest path from node to leaf
- height of tree: height of root
- time for Heapify(i) = O(height of subtree rooted at i)
- assume $n = 2^k - 1$ (a complete binary tree)

Building a heap: Analysis (3)

- For the n/2 nodes of height 1, heapify() requires at most 1 swap each.
- For the n/4 nodes of height 2, heapify() requires at most 2 swaps each.
- For the $n/2^i$ nodes of height i, heapify() requires at most i swaps each.
- So total number of swaps required is

$$
\begin{aligned}
T(n) &= O\left(\frac{n+1}{2} + \frac{n+1}{4}\cdot 2 + \frac{n+1}{8}\cdot 3 + \ldots + 1\cdot k\right) \\
&= O\left((n+1)\cdot \sum_{i=1}^{\lfloor \lg n \rfloor} \frac{i}{2^i}\right) \text{ since } \sum_{i=1}^{\lfloor \lg n \rfloor} \frac{i}{2^i} = \frac{1/2}{(1-1/2)^2} = 2 \\
&= O(n)
\end{aligned}
$$

Building a Heap: Analysis (4)

- How? By using the following "trick"

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x| < 1 \text{ //differentiate}$$

$$\sum_{i=1}^{\infty} i \cdot x^{i-1} = \frac{1}{(1-x)^2} \text{ //multiply by } x$$

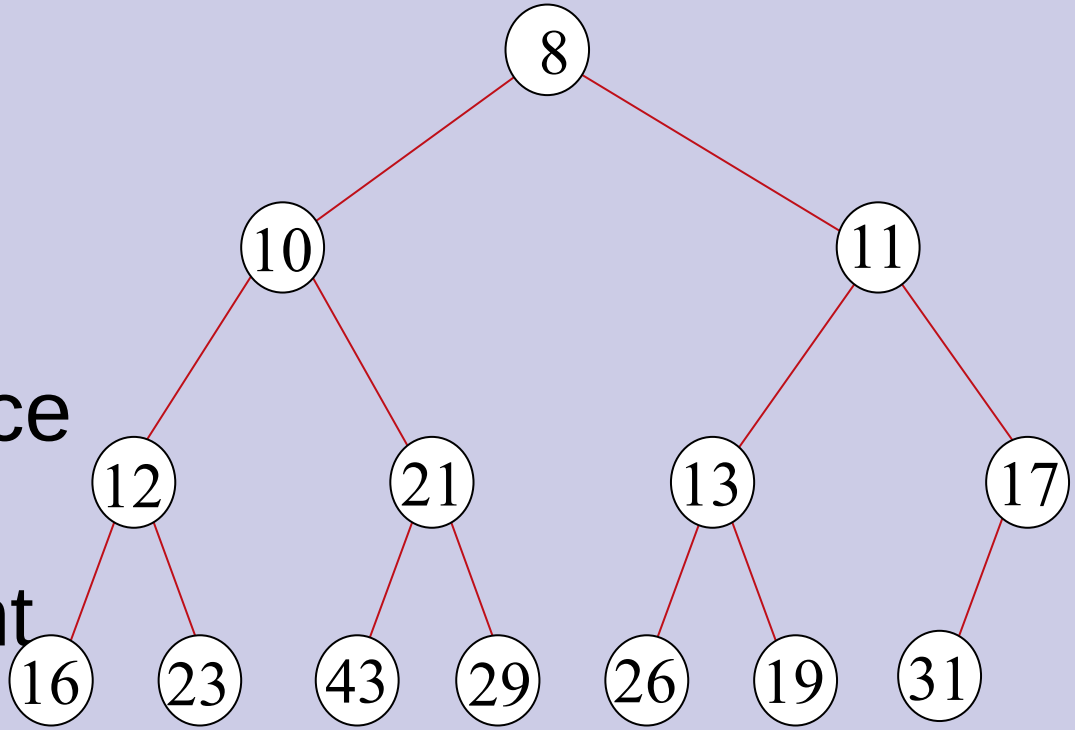
$$\sum_{i=1}^{\infty} i \cdot x^{i} = \frac{x}{(1-x)^2} \text{ //plug in } x = \frac{1}{2}$$

$$\sum_{i=1}^{\infty} \frac{i}{2^i} = \frac{1/2}{1/4} = 2$$

- Therefore Build-Heap time is O(*n*)

Heap Sort

- Create a heap.
- Do delete-min repeatedly till heap becomes empty.
- To do an in place sort, we move deleted element to end of heap.



Running times of heap operations

- Insert: O(log n)
- Heapify: O(log n)
- Find minimum: O(1)
- Delete-min: O(log n)
- Building a heap: O(n)
- Heap Sort: O(nlog n)