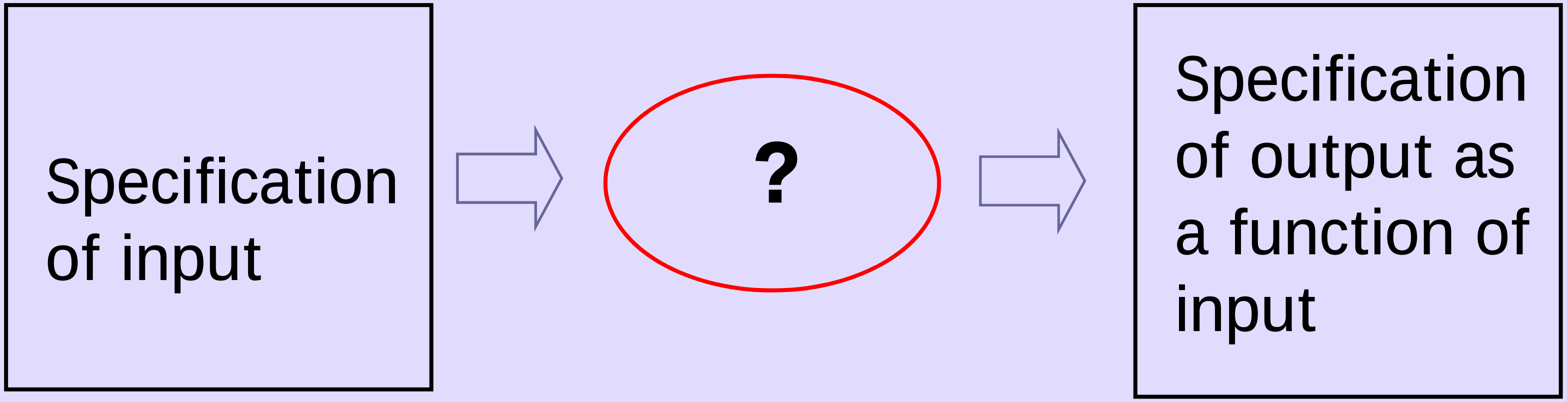
Data Structures and Algorithms

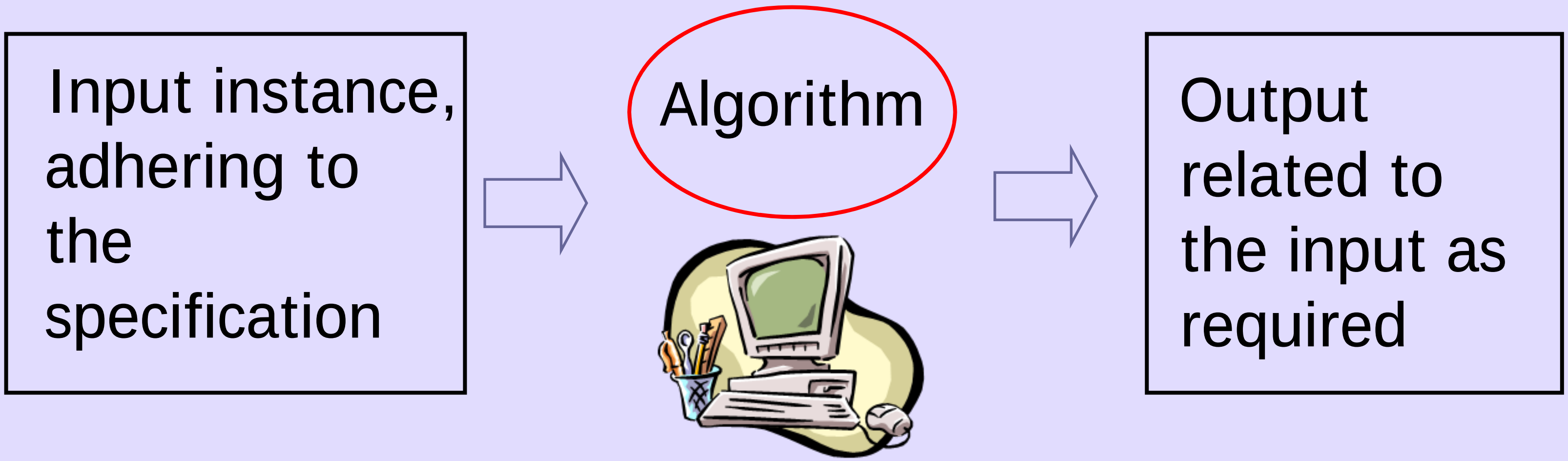
- Algorithm: Outline, the essence of a computational procedure, step-by-step instructions
- Program: an implementation of an algorithm in some programming language
- Data structure: **Organization** of data needed to solve the problem

Algorithmic problem



- Infinite number of input *instances* satisfying the specification. For eg: A sorted, non-decreasing sequence of natural numbers of non-zero, finite length:
  - 1, 20, 908, 909, 100000, 1000000000.
  - 3.

Algorithmic Solution



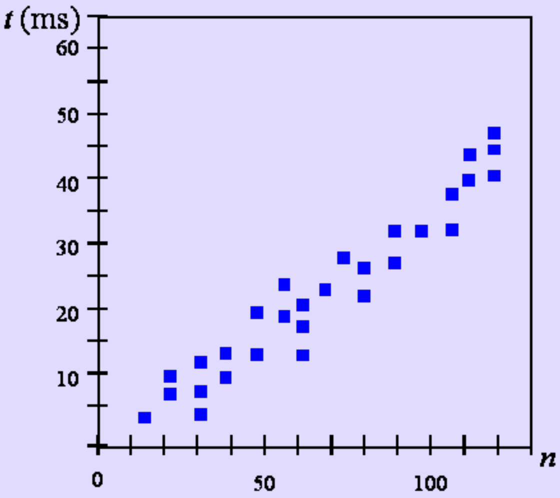
- Algorithm describes actions on the input instance
- Infinitely many correct algorithms for the same algorithmic problem

What is a Good Algorithm?

- Efficient:
  - Running time
  - Space used
- Efficiency as a function of input size:
  - The number of bits in an input number
  - Number of data elements (numbers, points)

Measuring the Running Time

How should we measure the running time of an algorithm?



Experimental Study

- Write a program that implements the algorithm
- Run the program with data sets of varying size and composition.
- Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time.

Limitations of Experimental Studies

- It is necessary to implement and test the algorithm in order to determine its running time.
- Experiments can be done only on a limited set of inputs, and may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments should be used.

Beyond Experimental Studies

We will develop a general methodology for analyzing running time of algorithms. This approach

- Uses a high-level description of the algorithm instead of testing one of its implementations.
- Takes into account all possible inputs.
- Allows one to evaluate the efficiency of any algorithm in a way that is independent of the hardware and software environment.

Pseudo-Code

- A mixture of natural language and high-level programming concepts that describes the main ideas behind a generic implementation of a data structure or algorithm.
- Eg: **Algorithm** arrayMax(A, n):

    Input*:* An array A storing n integers.

    Output*:* The maximum element in A.

```
currentMax ← A[0]
for  i ← 1 to n-1 do
   if currentMax < A[i] then currentMax ← A[i]
return currentMax
```

Pseudo-Code

It is more structured than usual prose but less formal than a programming language

- Expressions:
  - use standard mathematical symbols to describe numeric and boolean expressions
  - use $\leftarrow$ for assignment ("=" in Java)
  - use = for the equality relationship ("==" in Java)
- Method Declarations:
  - **Algorithm** name(param1**,** param2)

Pseudo Code

- Programming Constructs:
  - decision structures: **if ... then ... [else ...]**
  - while-loops: **while ... do**
  - repeat-loops: **repeat ... until ...**
  - for-loop: **for ... do**
  - array indexing: **A[i], A[i,j]**
- Methods:
  - calls: object method(args)
  - returns: **return** value

Analysis of Algorithms

- **Primitive Operation:** Low-level operation independent of programming language. Can be identified in pseudo-code. For eg:
  - Data movement (assign)
  - Control (branch, subroutine call, return)
  - arithmetic an logical operations (e.g. addition, comparison)
- By inspecting the pseudo-code, we can count the number of primitive operations executed by an algorithm.

Example: Sorting

INPUT

sequence of numbers

$a_1, a_2, a_3, \ldots, a_n$

2 5 4 10 7

OUTPUT

a permutation of the sequence of numbers

$b_1, b_2, b_3, \ldots, b_n$

2 4 5 7 10

Correctness (requirements for the output)

For any given input the algorithm halts with the output:

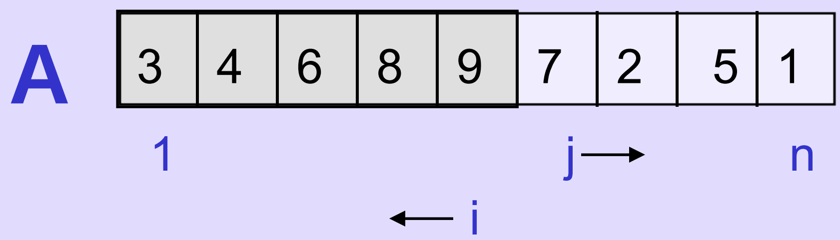
- $b_1 < b_2 < b_3 < \ldots < b_n$
- $b_1, b_2, b_3, \ldots, b_n$ is a permutation of $a_1, a_2, a_3, \ldots, a_n$

Running time

Depends on

- number of elements (n)
- how (partially) sorted they are
- algorithm

Insertion Sort



Strategy

- Start "empty handed"
- Insert a card in the right position of the already sorted hand
- Continue until all cards are inserted/sorted

INPUT: A[1..n] – an array of integers
OUTPUT: a permutation of A such that $A[1] \leq A[2] \leq \ldots \leq A[n]$

```
for j←2 to n do
    key ← A[j]
    Insert A[j] into the sorted sequence
    A[1..j-1]
    i←j-1
    while i>0 and A[i]>key
        do A[i+1]←A[i]
           i--
    A[i+1]←key
```

Analysis of Insertion Sort

	cost	times
for j←2 **to** *n* **do**	c_1	n
key←A[j]	c_2	n-1
Insert A[j] into the sorted sequence A[1..j-1]	0	n-1
i←j-1	c_3	n-1
while i>0 **and** A[i]>key	c_4	$\sum_{j=2}^{n} t_j$
do A[i+1]←A[i]	c_5	$\sum_{j=2}^{n} (t_j - 1)$
i--	c_6	$\sum_{j=2}^{n} (t_j - 1)$
A[i+1] ← key	c_7	n-1

$$\text{Total time} = n(c_1+c_2+c_3+c_7) + \sum_{j=2}^{n} t_j \; (c_4+c_5+c_6) - (c_2+c_3+c_5+c_6+c_7)$$

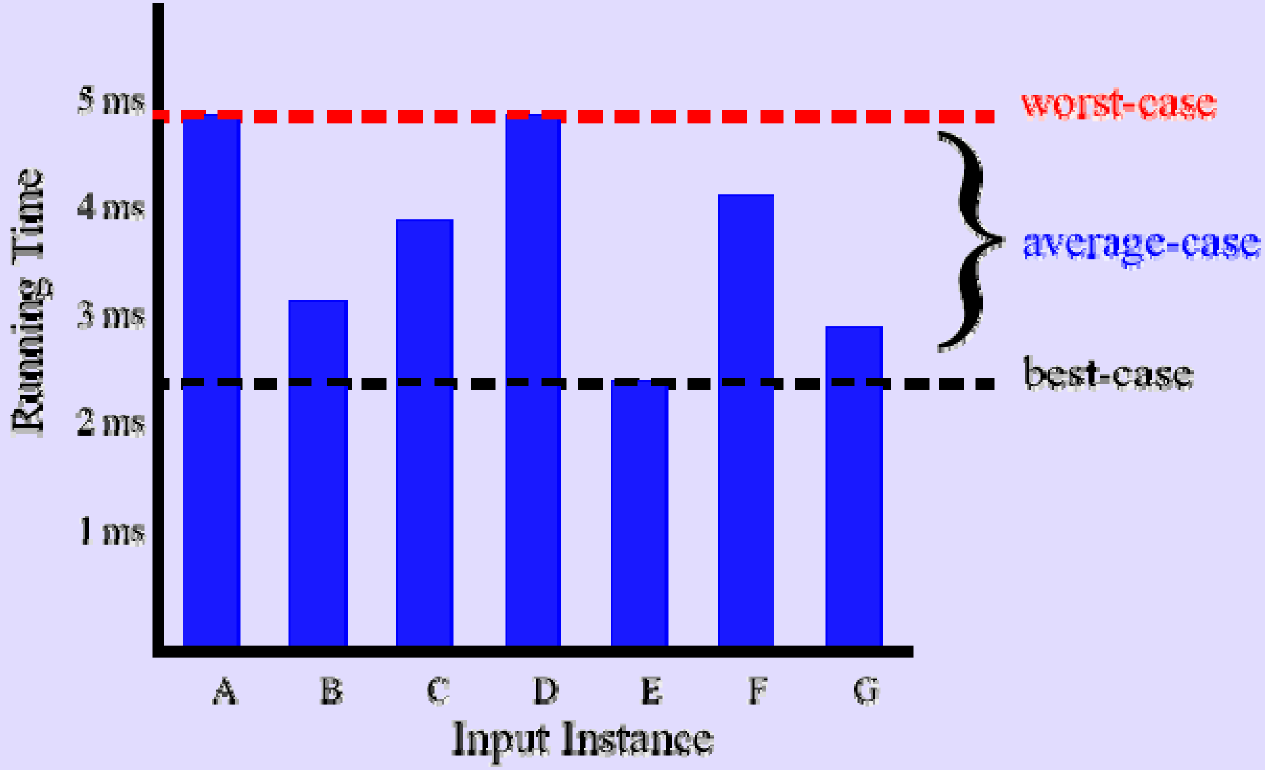
Best/Worst/Average Case

Total time = $n(c_1+c_2+c_3+c_7) + \sum^{n}_{j=2} t_j \ (c_4+c_5+c_6)$
$- (c_2+c_3+c_5+c_6+c_7)$

- **Best case**: elements already sorted; $t_j=1$, running time = f(n), i.e., *linear* time.
- **Worst case**: elements are sorted in inverse order; $t_j=j$, running time = $f(n^2)$, i.e., *quadratic* time
- **Average case**: $t_j=j/2$, running time = $f(n^2)$, i.e., *quadratic* time

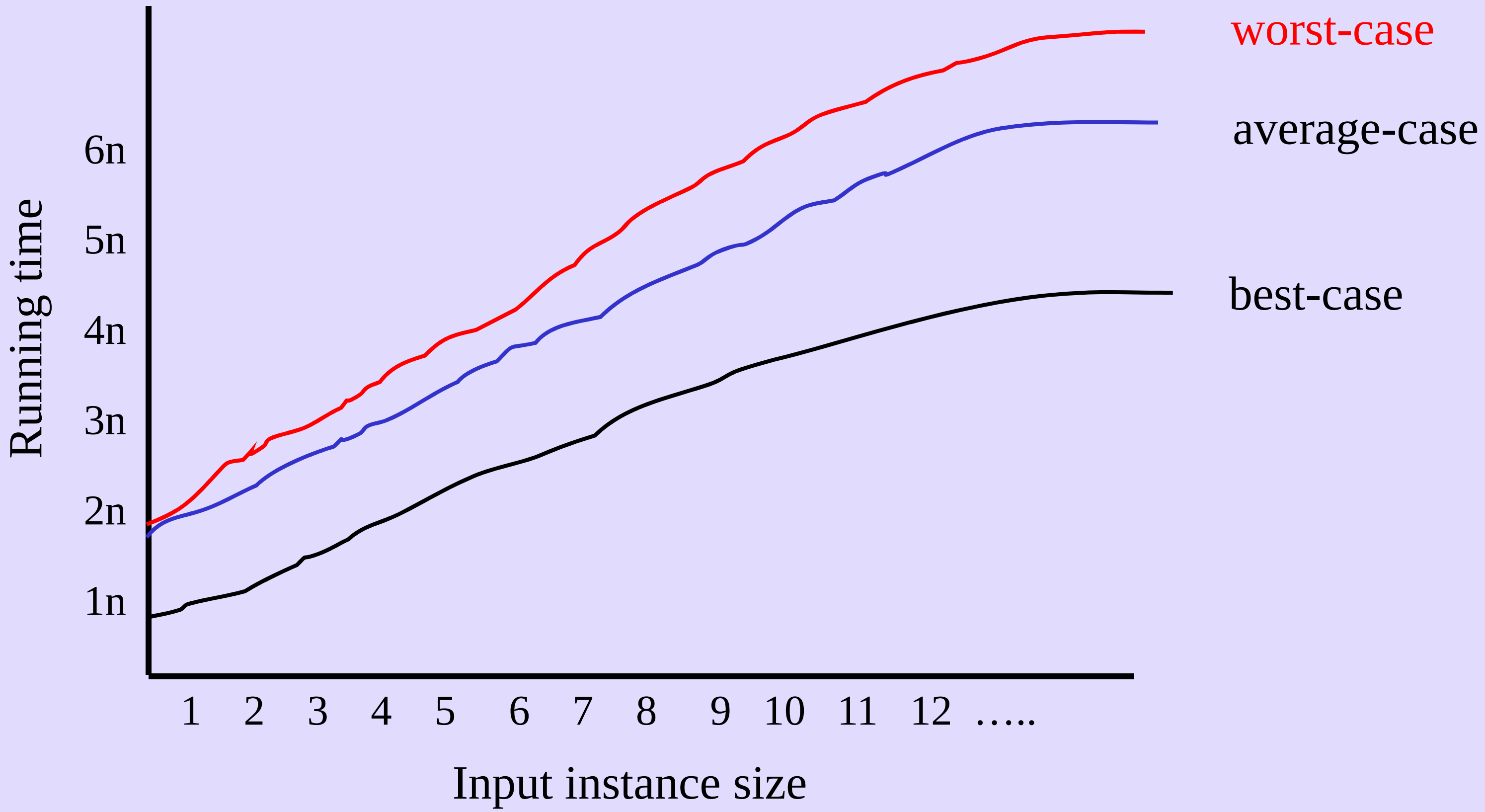
Best/Worst/Average Case (2)

- For a specific size of input n, investigate running times for different input instances:



Best/Worst/Average Case (3)

For inputs of all sizes:



Best/Worst/Average Case (4)

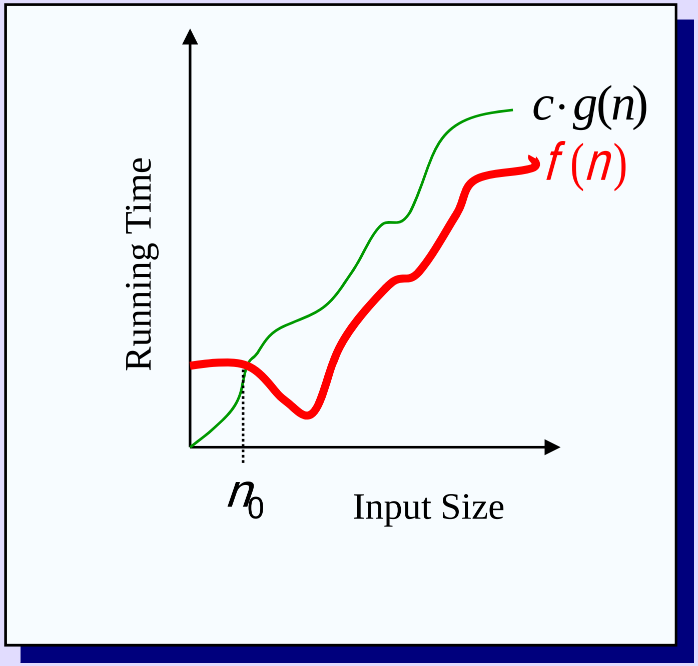
- **Worst case** is usually used: It is an upper-bound and in certain application domains (e.g., air traffic control, surgery) knowing the **worst-case** time complexity is of crucial importance
- For some algorithms **worst case** occurs fairly often
- **Average case** is often as bad as the **worst case**
- Finding **average case** can be very difficult

Asymptotic Analysis

- Goal: to simplify analysis of running time by getting rid of ”details”, which may be affected by specific implementation and hardware
  - like “rounding”: 1,000,001 $\approx$ 1,000,000
  - $3n^2 \approx n^2$
- Capturing the essence: how the running time of an algorithm increases with the size of the input *in the limit*.
  - Asymptotically more efficient algorithms are best for all but small inputs

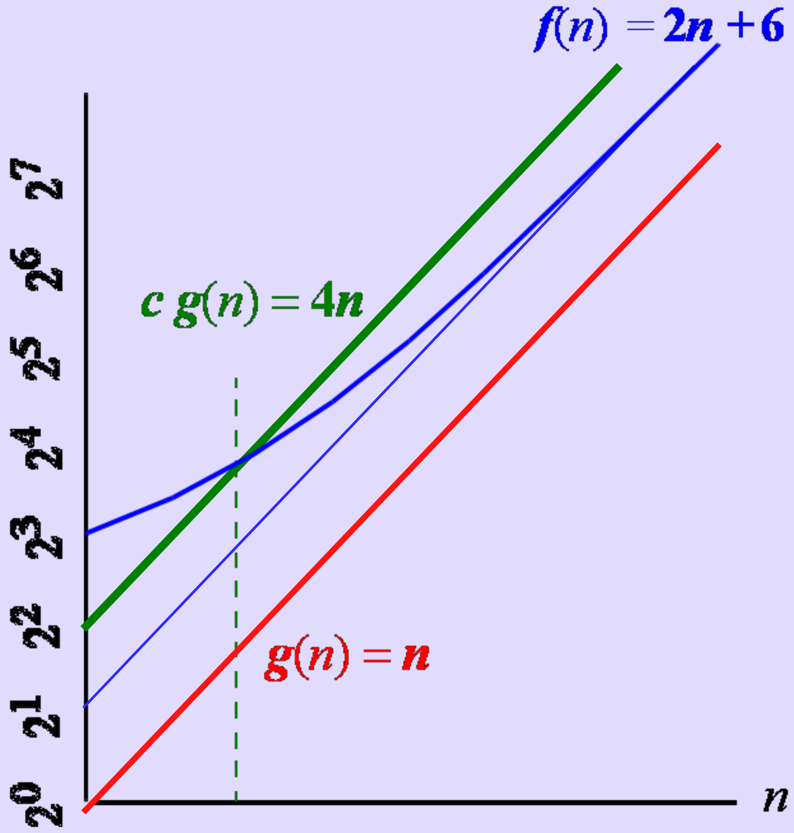
Asymptotic Notation

- The "big-Oh" *O*-Notation
  - asymptotic upper bound
  - f(n) is O(g(n)), if there exists constants *c* and n_0, *s.t.* $\mathbf{f(n) \leq c\, g(n)}$ for $n \geq n_0$
  - f(n) and g(n) are functions over non-negative integers
- Used for *worst-case* analysis



Example

For functions f(n) and g(n) there are positive constants c and n_0 such that: $f(n) \leq c\, g(n)$ for $n \geq n_0$



conclusion:

2n+6 is O(n).

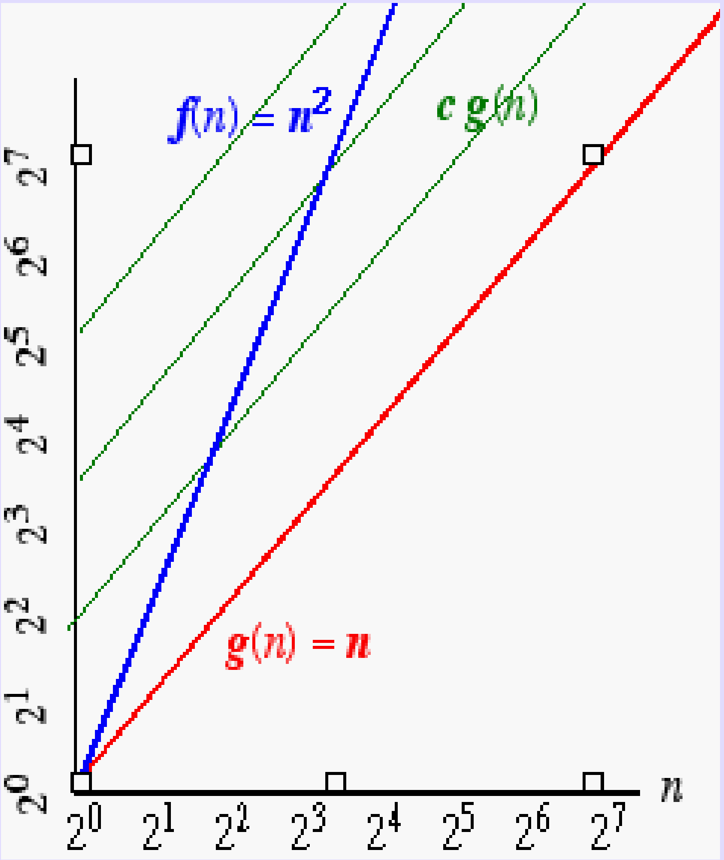
Another Example

On the other hand…

n^2 is not O(n) because there is no c and n_0 such that:

$$n^2 \le cn \text{ for } n \ge n_0$$

The graph to the right illustrates that no matter how large a c is chosen there is an n big enough that $n^2 > cn$)



Asymptotic Notation

- Simple Rule: Drop lower order terms and constant factors.
  - $50\ n \log n$ is $O(n \log n)$
  - $7n - 3$ is $O(n)$
  - $8n^2 \log n + 5n^2 + n$ is $O(n^2 \log n)$
- Note: Even though ($50\ n \log n$) is $O(n^5)$, it is expected that such an approximation be of as small an order as possible

Asymptotic Analysis of Running Time

- Use *O*-notation to express number of primitive operations executed as function of input size.
- Comparing asymptotic running times
  - an algorithm that runs in $O(n)$ time is better than one that runs in $O(n^2)$ time
  - similarly, $O(\log n)$ is better than $O(n)$
  - hierarchy of functions: $\log n < n < n^2 < n^3 < 2^n$
- Caution! Beware of very large constant factors. An algorithm running in time 1,000,000 n is still $O(n)$ but might be less efficient than one running in time $2n^2$, which is $O(n^2)$

Example of Asymptotic Analysis

Algorithm prefixAverages1(X):

Input: An n-element array X of numbers.

Output: An n-element array A of numbers such that A[i] is the average of elements X[0], ... , X[i].

for i ← 0 **to** n-1 **do**

  a ← 0

  for j ← 0 **to** i **do**

    a ← a + X[j] ← 1 step

  A[i] ← a/(i+1)

return array A

i iterations with i=0,1,2...n-1

n iterations

Analysis: running time is $O(n^2)$

A Better Algorithm

Algorithm prefixAverages2(X):

Input: An *n*-element array X of numbers.

Output: An *n*-element array A of numbers such that A[*i*] is the average of elements X[0], ... , X[*i*].

s ← 0

for i ← 0 **to** n **do**

    s ← s + X[i]

    A[i] ← s/(i+1)

return array A

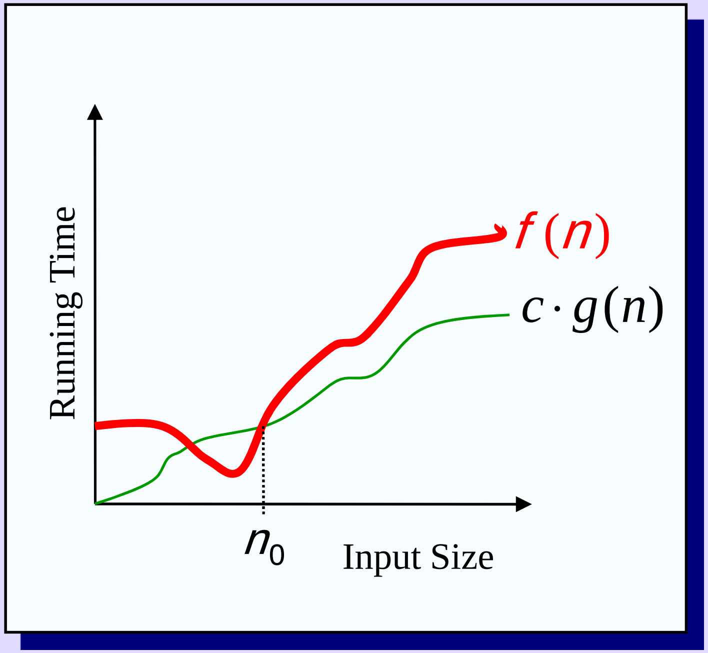
Analysis: Running time is O(n)

Asymptotic Notation *(terminology)*

- Special classes of algorithms:
  - Logarithmic: $O(\log n)$
  - Linear: $O(n)$
  - Quadratic: $O(n^2)$
  - Polynomial: $O(n^k)$, $k \geq 1$
  - Exponential: $O(a^n)$, $a > 1$
- "Relatives" of the Big-Oh
  - Ω (f(n)): Big Omega -asymptotic *lower* bound
  - Θ (f(n)): Big Theta -asymptotic *tight* bound

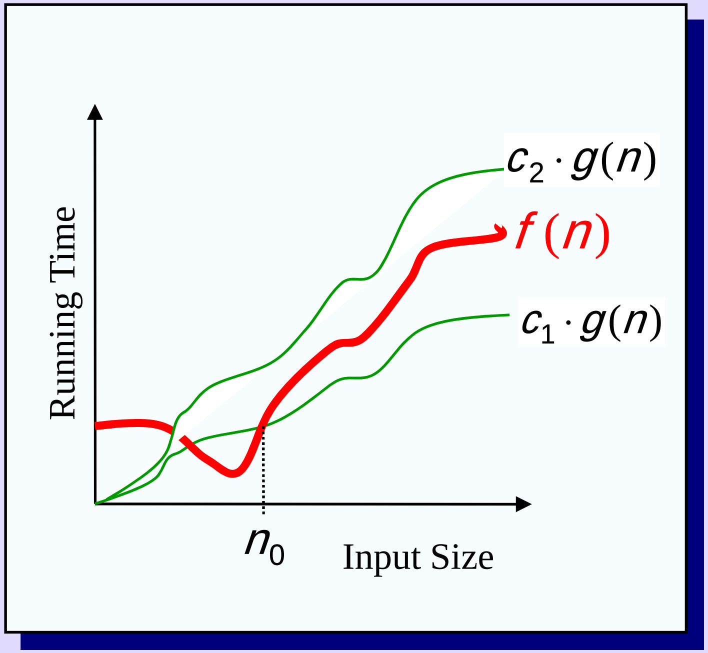
Asymptotic Notation

- The "big-Omega" Ω–Notation
  - asymptotic lower bound
  - $f(n)$ is $\Omega(g(n))$ if there exists constants c and n_0, *s.t.* $c\, g(n) \leq f(n)$ for $n \geq n_0$
- Used to describe *best-case* running times or lower bounds for algorithmic problems
  - E.g., lower-bound for searching in an unsorted array is $\Omega(n)$.



Asymptotic Notation

- The "big-Theta" Θ–Notation
  - asymptotically tight bound
  - $f(n) = \Theta(g(n))$ if there exists constants c_1, c_2, and n_0, *s.t.* $\mathbf{c_1\, g(n) \le f(n) \le c_2\, g(n)}$ for $n \ge n_0$
- $f(n)$ is $\Theta(g(n))$ if and only if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$
- $O(f(n))$ is often misused instead of $\Theta(f(n))$



Asymptotic Notation

Two more asymptotic notations

- "Little-Oh" notation *f(n)* is *o(g(n))* non-tight analogue of Big-Oh
  - For every *c*, there should exist n_0, s.t. $\mathbf{f(n) \leq c\ g(n)}$ for $n \geq n_0$
  - Used for **comparisons** of running times. If *f(n)=o(g(n))*, it is said that *g(n) dominates f(n).*
- "Little-omega" notation *f(n)* is $\omega(g(n))$ non-tight analogue of Big-Omega

Asymptotic Notation

- Analogy with real numbers
  - $f(n) = O(g(n)) \cong f \leq g$
  - $f(n) = \Omega(g(n)) \cong f \geq g$
  - $f(n) = \Theta(g(n)) \cong f = g$
  - $f(n) = o(g(n)) \cong f < g$
  - $f(n) = \omega(g(n)) \cong f > g$
- Abuse of notation: $f(n) = O(g(n))$ actually means $f(n) \in O(g(n))$

Comparison of Running Times

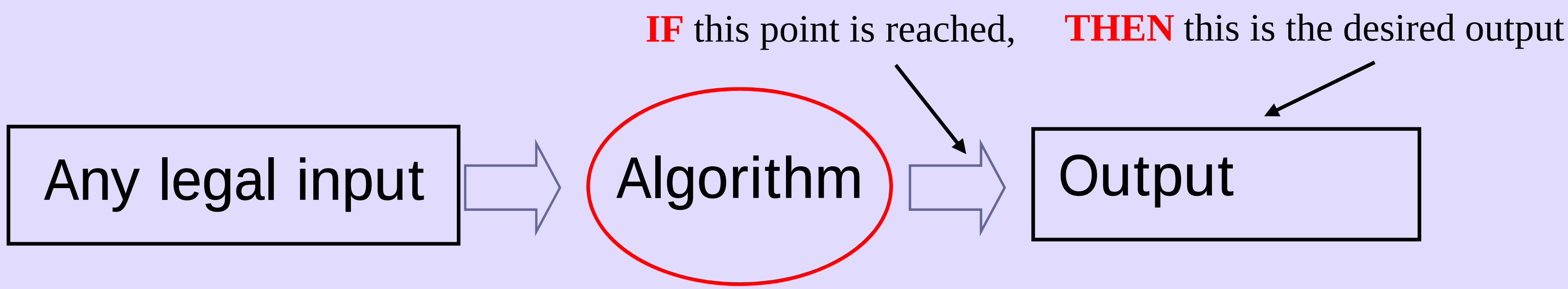
Running Time	Maximum problem size (n)		
	1 second	1 minute	1 hour
$400n$	2500	150000	9000000
$20n \log n$	4096	166666	7826087
$2n^2$	707	5477	42426
n^4	31	88	244
2^n	19	25	31

Correctness of Algorithms

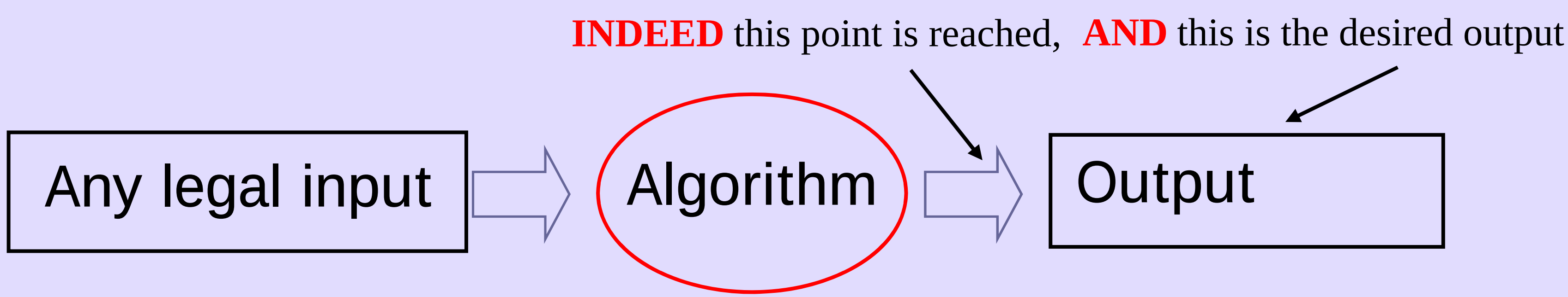
- The algorithm is *correct* if for any legal input it terminates and produces the desired output.
- Automatic proof of correctness is not possible
- But there are practical techniques and rigorous formalisms that help to reason about the correctness of algorithms

Partial and Total Correctness

- Partial correctness



- Total correctness



Assertions

- To prove correctness we associate a number of **assertions** (statements about the state of the execution) with specific checkpoints in the algorithm.
  - E.g., *A*[1], …, *A*[k] form an increasing sequence
- **Preconditions** – assertions that must be valid *before* the execution of an algorithm or a subroutine
- **Postconditions** – assertions that must be valid *after* the execution of an algorithm or a subroutine

Loop Invariants

- **Invariants** – assertions that are valid any time they are reached (many times during the execution of an algorithm, e.g., in loops)
- We must show three things about loop invariants:
  - **Initialization** – it is true prior to the first iteration
  - **Maintenance** – if it is true before an iteration, it remains true before the next iteration
  - **Termination** – when loop terminates the invariant gives a useful property to show the correctness of the algorithm

Example of Loop Invariants (1)

- **Invariant**: *at the start of each **for** loop, A[1…j-1] consists of elements originally in A[1…j-1] but in sorted order*

```
for j ← 2 to length(A)
   do key ← A[j]
      i ← j-1
      while i>0 and A[i]>key
        do A[i+1] ← A[i]
           i--
      A[i+1] ← key
```

Example of Loop Invariants (2)

- **Invariant**: *at the start of each **for** loop, A[1…j-1] consists of elements originally in A[1…j-1] but in sorted order*

```
for j ← 2 to length(A)
   do key ← A[j]
      i ← j-1
      while i>0 and A[i]>key
        do A[i+1]← A[i]
           i--
      A[i+1] ← key
```

- **Initialization**: *j = 2,* the invariant trivially holds because *A*[1] is a sorted array ☺

Example of Loop Invariants (3)

- **Invariant**: *at the start of each* ***for*** *loop, A[1…j-1] consists of elements originally in A[1…j-1] but in sorted order*

```
for j ← 2 to length(A)
   do key ← A[j]
      i ← j-1
      while i>0 and A[i]>key
        do A[i+1] ← A[i]
           i--
      A[i+1] ← key
```

- **Maintenance**: the inner **while** loop moves elements $A[j-1]$, $A[j-2]$, ..., $A[j-k]$ one position right without changing their order. Then the former $A[j]$ element is inserted into k-th position so that $A[k-1] \le A[k] \le A[k+1]$.

$A[1...j-1]$ sorted + $A[j] \rightarrow A[1...j]$ sorted

Example of Loop Invariants (4)

- **Invariant**: *at the start of each **for** loop, A[1…j-1] consists of elements originally in A[1…j-1] but in sorted order*

```
for j ← 2 to length(A)
   do key ← A[j]
      i ← j-1
      while i>0 and A[i]>key
        do A[i+1] ← A[i]
           i--
      A[i+1] ← key
```

- **Termination**: the loop terminates, when *j=n+1*. Then the invariant states: *"A[1...n] consists of elements originally in A[1...n] but in sorted order"* ☺

Math You Need to Review

- Properties of logarithms:

  $\log_b(xy) = \log_b x + \log_b y$

  $\log_b(x/y) = \log_b x - \log_b y$

  $\log_b x^a = a \log_b x$

  $\log_b a = \log_x a / \log_x b$

- Properties of exponentials:

  $a^{(b+c)} = a^b a^c$; $a^{bc} = (a^b)^c$

  $a^b / a^c = a^{(b-c)}$; $b = a^{\log_a b}$

- Floor: $\lfloor x \rfloor$ = the largest integer $\leq x$
- Ceiling: $\lceil x \rceil$ = the smallest integer $\geq x$

Math Review

- Geometric progression
  - given an integer n_0 and a real number $0 < a \neq 1$

$$\sum_{i=0}^{n} a^i = 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

  - geometric progressions exhibit exponential growth
- Arithmetic progression

$$\sum_{i=0}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n^2 + n}{2}$$

Summations

- The running time of insertion sort is determined by a nested loop

```
for j←2 to length(A)
   key←A[j]
   i←j-1
   while i>0 and A[i]>key
      A[i+1]←A[i]
      i←i-1
   A[i+1]←key
```

- Nested loops correspond to summations

$$\sum_{j=2}^{n} (j-1) = O(n^2)$$

Proof by Induction

- We want to show that property P is true for all integers $n \geq n_0$
- **Basis**: prove that P is true for n_0
- **Inductive step**: prove that if P is true for all k such that $n_0 \leq k \leq n - 1$ then P is also true for n
- Example $$S(n) = \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \text{ for } n \geq 1$$
- Basis $$S(1) = \sum_{i=0}^{1} i = \frac{1(1+1)}{2}$$

Proof by Induction (2)

- Inductive Step

$$S(k) = \sum_{i=0}^{k} i = \frac{k(k+1)}{2} \text{ for } 1 \leq \text{k} \leq n-1$$

$$S(n) = \sum_{i=0}^{n} i = \sum_{i=0}^{n-1} i + n = S(n-1) + n =$$

$$= (n-1)\frac{(n-1+1)}{2} + n = \frac{(n^2 - n + 2n)}{2} =$$

$$= \frac{n(n+1)}{2}$$